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# Non linear treatment of isotropic flat plates with large deflections by finite deformation theory.

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NON LINEAR TREATMENT OF
ISOTROPIC FLAT PLATES WITH
LARGE DEFLECTIONS
BY FINITE DEFORMATION THEORY

By

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# BY

# FINITE DEFORMATION THEORY

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By

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ABSTRACT

In this thesis, large deflection plate compatibility and equilibrium differential equations are derived using finite deformation theory including the non-linear five constant stress-strain relationship. In addition, the assumptions of classical large deflection plate problems under classical elasticity theory are discussed in connection with the more general approach proposed by this thesis.

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# NOTATION

$a, b, c$ = Curvilinear Coordinates

$x, y, z$ = Cartesian Coordinates

$X, Y, Z$ = Body Forces in the x, y, and z directions

$A, B, C, D$ = Coefficients =

$$A = \frac{2G\nu}{1-\nu}$$

$$B = \frac{m}{2}\,\frac{1-2\nu}{1-\nu}$$

$$C = 2G$$

$$D = -\frac{n}{4}$$

$D$ also = Flexural Rigidity = $\dfrac{Et^3}{12(1-\nu^2)}$

$E$ = Young's Modulus = Modulus of Elasticity

$E_3$ = Unit Matrix = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$F$ = Stress Function defined such that

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} \quad ; \quad \sigma_y = \frac{\partial^2 F}{\partial x^2} \quad ; \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x\,\partial y}$$

$G$ = Shear Modulus

$I_1$ = First Invariant of Strain Tensor

$I_2$ = Second Invariant of Strain Tensor

$I_3$ = Third Invariant of Strain Tensor

$\mathcal{J}_{1c}$ = First Invariant of Curvature Tensor

$\mathcal{J}_{2c}$ = Second Invariant of Curvature Tensor

$\mathcal{J}_{1LD}$ = First Invariant of Large Deflection Tensor

$J$ = Jacobian = $\dfrac{\partial(x,y,z)}{\partial(a,b,c)} = \begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{pmatrix}$

$J^*$ = Transpose of Jacobian = $\begin{pmatrix} \frac{\partial}{\partial a} \\ \frac{\partial}{\partial b} \\ \frac{\partial}{\partial c} \end{pmatrix} \begin{pmatrix} x & y & z \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{pmatrix}$

l, m, n = Third Order Elastic Constants

$M = J^* J = 2\eta + E_3$

Also

$$M = \text{Moment Tensor} = \begin{pmatrix} M_{aa} & -M_{ab} & -Q_{ac} \\ M_{ba} & M_{bb} & -Q_{bc} \\ -Q_{ac} & -Q_{bc} & -\frac{pc^2}{2} \end{pmatrix}$$

M subscripted $(M_x, M_y, M_{xy})$ = Moment per unit length

P = Pressure

Q subscripted $(Q_x, Q_y)$ = Shear per unit length

R = Rotational Matrix

$$\mathcal{R} = \text{Curvature Tensor} = \begin{pmatrix} -\frac{1}{r_{aa}} & \frac{1}{r_{ab}} & \frac{1}{r_{ac}} \\ \frac{1}{r_{ba}} & -\frac{1}{r_{bb}} & \frac{1}{r_{bc}} \\ \frac{1}{r_{ca}} & \frac{1}{r_{cb}} & -\frac{1}{r_{cc}} \end{pmatrix}$$

r subscripted $(r_x, r_y, r_{xy})$ = Radius of Curvature

S subscripted = Area of Elemental Side After Deformation

$S^*$ subscripted = Area of Elemental Side Before Deformation

$$T = \text{Stress Tensor} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

V = Volume of Element Before Deformation

$V^*$ = Volume of Element After Deformation

$\rho_a$ = Initial Density

$\rho_x$ = Deformed Density

$\frac{\rho_x}{\rho_a}$ = Compression Ratio = $\frac{1}{\det J}$

$$\eta = \text{Strain Tensor} = \begin{pmatrix} e_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & e_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & e_z \end{pmatrix}$$

$\mu$ = Lame's Constant = $G$

$\lambda$ = Lame's Constant = $\dfrac{\nu E}{(1+\nu)(1-2\nu)}$ = $\dfrac{2G\nu}{1-2\nu}$

$\nu$ = Poisson's Ratio

$\sigma$ = Tensile or Compressive Stress (Subscripted to indicate orientation)

$\tau$ = Shear Stress (subscripted)

$e$ = Strain normal to face of element (subscripted)

$\gamma$ = Shear Strain (subscripted)

$\epsilon$ = Term indicating small quantity $\ll 1$ for purposes of determination of order of magnitude.

$\phi(\eta) = \rho_a \psi$ = Energy of deformation / unit initial volume

$\nabla^2$ = Laplacian = $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}\right)$

$$\nabla^4 = \left(\frac{\partial^4}{\partial x^4} + \frac{2\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right)$$

## INTRODUCTION

The approach to the large deflection plate problem proposed by this thesis, as suggested by the title, is one utilizing non-linear elasticity theory, or theory dealing with finite deformation of elastic solids. In reference [1] Francis D. Murnaghan provides an excellent discussion of the theory, and derives the basic stress-strain relationship used in this thesis (see appendix B). The use of this basic non-linear stress-strain relationship, sometimes called Murnaghan's law [2], places the scope of this thesis more appropriately into the field of rheology rather than structures per se (see Figure 1).

In actuality, predicting how materials will behave in response to forces lies in the domain of rheology - literally the study of flow. The Society of Rheology uses the words of Heraclitus, παντα ῥει (everything flows), as its motto, and books on rheology quote this philosophical conviction in their introductory pages. [3]. Because rheology deals with the flow of matter under the action of forces, by an obvious extension it also includes all deformations of materials by forces. Thus,

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[1] References are listed beginning on page 66.

[2] Novozhilov states "The elastic law corresponding to it (the five constant stress-strain theory) is ordinarily called Murnaghan's law, although it was actually first proposed much earlier by Voigt in 1893. The first attempt to examine the stress-strain relation in a form different from Hooke's law was made by Bulffinger in a paper published in the works of the Russian Academy of Sciences in 1729" pp. 127.

as shown in Figure 1, rheological theories have as their model the classical theories of elasticity and hydrodynamics in which it is assumed that a material, whether solid or liquid, deforms linearly in response to a force. From the classical models have grown more and more general theories, first by abandoning classical distinctions between a solid and a liquid, and then by abandoning the requirement that the response be linear. The most general theory, as shown in Figure (1), is the theory of nonlinear visco-elasticity which includes all other theories as special cases. A theory of non linear visco-inelasticity exists (shown in Figure 1 dashed) but is of historical interest only [3].

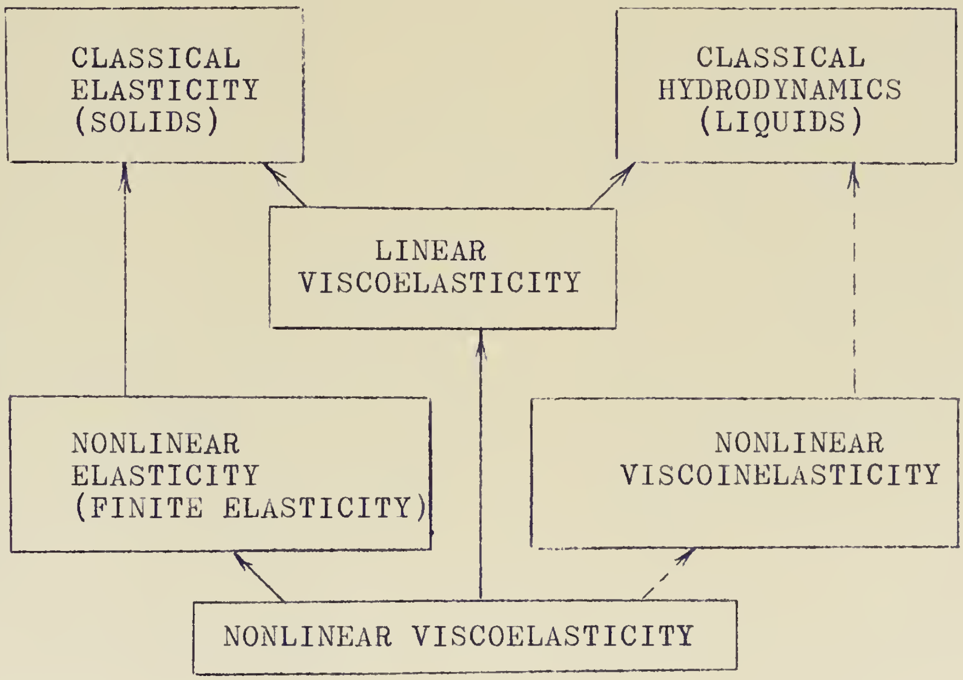



FIGURE 1

(From "Deformation and Flow" by Charles J. Lynch as appeared in <u>International Science and Technology</u> January 1966)

Since the end of World War II there has been a considerable change in the approach to rheology and to the formulation of rheological theories. The origin of this change is two-fold. First, certain flow phenomena, observed first in flame-thrower fuels and later in a wide variety of other fuels, were in direct opposition to classical hydrodynamic theory. The other factor, which was perhaps of greater significance in its contribution to the changing approach to rheology, at least to those primarily interested in structures, was the development of a continuum theory for elastic materials.

Although the theory of finite deformation of an elastic solid applies equally to all elastic materials, most of the research accomplished to date has been in the rubber industry. The theory as applied to metallic structures has only recently become of practical interest with the increased emphasis on limit design, and the more stringent requirements of high performance aircraft and missiles, space craft, and deep submergence pressure hulls. In the majority of experimental research conducted so far, rubber is generally used because it has a large elastic range and its nonlinearities are therefore more easily measured.

Vulcanized rubber differs from other elastic materials in the extent of its elastic deformation. A rubber rod or strip may be stretched to four or five times its initial length without suffering permanent deformation. When rubber is subjected to shear deformation and the corresponding shearing force is measured,

the relation between the shear strain and the shearing force is found to be approximately linear, just as with most elastic solids. On the other hand, when rubber is stretched by tensile forces, the relation between the tensile force and the elongation shows considerable departure from linear behavior.

While this may at first sight appear surprising, it can in fact be demonstrated mathematically that a material having a linear shear relationship cannot possibly have a linear tensile relationship. From this we can conclude that a true Hookean solid is only a convenient fiction - an elastic material cannot possibly have a linear relationship in both tension and shear as assumed in classical elasticity theory. The classical theory is only valid because the departure from linearity does not show up when the deformations are small - even a curved line appears straight if one is only concerned with a little piece of it. [3].

In finite elasticity theory, the material properties are characterized by an expression for the energy of deformation per unit initial volume, ($\phi(\eta)$ ), [1] (see appendix B). This strain energy function and the associated material constants have been determined by experiment for various vulcanized rubbers and the actual manner in which stored energy depends on the deformation has been determined from them. Using this experimentally determined stored energy, the results of other experiments with vulcanized rubber have been predicted with considerable accuracy, providing an excellent verification of the theory [3]. Although the material constants of interest to the structural engineer

(modulus of elasticity and shear modulus plus the three additional third order elastic constants shown in appendix B) have been evaluated for several structural materials as shown in Table I, the numerical values show a wide variation between investigators with a wide spread for some materials and much more work is required.

TABLE I

THIRD ORDER ELASTIC CONSTANTS ( ± numbers are the range of values)

| Reference and Material | E psi X $10^{-6}$ Relative Value | l psi X $10^{-6}$ | m psi X $10^{-6}$ | n psi X $10^{-6}$ |
|---|---|---|---|---|
| Hughes and Kelly Armco Iron | 27.5 | − 34.8 ± 6.5 | − 10.3 ± 7.0 | + 110 ± 110 |
| Hughes and Kelly Pyrex | 2.0 | + 1.4 ± 4.0 | + 9.2 ± 5.0 | + 42 ± 35 |
| Seeger and Buck Copper | 17.0 | − 22.6 ± 10 | − 88.0 ± 1.3 | − 225 ± 3 |
| Seeger and Buck Iron | 29.5 | − 24.3 ± 6 | − 110 ± 1.5 | − 215 ± 1.5 |
| Smith .6 Carbon Steel | 21.1 | − 46 | − 60 | − 67 |
| Smith Austenitic Steel | 18.3 | − 53.5 | − 75.2 | − 40.0 |
| Crecraft Nickel-Steel | 17.4 | − 4.6 | − 59 | − 73.0 |
| Rollins etal. 6016-T6 Aluminum | ——— | ——— | ——— | − 31.2 |
| Borg - Calculated values based on $\nu$ = .25 | 30.0 | ——— | − 45 | − 34.2 |

Data as reported in reference [6]

Finite elasticity theory has successfully predicted a number of effects that are not predicted by classical theory. For example, experimental observations of a rod of circular cross section subjected to simple torsion made by Polynting in 1909 with steel wires and in 1913 with vulcanized rubber rod showed that the rod will not only twist as predicted by classical theory, but will also elongate. As is often the case when knowledge available in the literature remains unused for some time, these results seem to have been forgotten until recently when the application of finite deformation theory predicted this result [3].

That Murnaghan's finite deformation theory [1] should be applied to plates and shells in the large deflection region has been proposed verbally by S. F. Borg for some time. In reference [4], Borg, Hoppe and Kopchinski, and in references [5], [6], [7] and [8] Borg discusses large deflection of plates, including the derivation of plate compatibility equations from a finite deformation theory approach. This thesis is a continuation and extension of Borg's work as applied to plates.

## GENERAL DISCUSSION OF NON LINEAR ELASTICITY

Non linearity is introduced into the theory of elasticity dealing with isotropic materials in three ways. It can be introduced through the generalized strain tensor or strain displacement relation (appendix A), through the stress-strain relations (appendix B) or through the equations of equilibrium of a volume element of the body (appendix C).

In the equations of equilibrium and the strain tensor relations, the retention of non-linear terms is conditioned by geometric considerations, or the necessity of considering the angles of rotation in determining dimensional changes of line elements and in formulating the conditions of equilibrium of a volume element. However, in the stress-strain relation, non-linear terms appear if the strain exceeds in magnitude, physical constants characteristic of the material and referred to as the limits of proportionality. It should be noted that the above discussion does not presuppose a linear elasticity origin. Actually, linear theory is developed by successively ignoring those terms or quantities which give non-linear characteristics to the resultant equations. For many materials, e.g. aluminum, the limit of proportionality can be quite low or non existant, and a finite deformation, continuum, theory is required where deformations are sufficiently large as to invalidate linear elasticity theory (Hooke's law).

Thus, two types of non-linearity must be considered in large deflection problems, geometrical and physical. Since, in

general, smallness of angles of rotation of the body as a whole does not imply the smallness of elongation or shear strain of an infinitesimal element of volume of the body (and conversely), the geometrical and physical non-linearity can be regarded as independent of each other. This independence is useful for purposes of classifying problem types and for purposes of making reasonable assumptions to simplify the problem to enable an engineering solution.

As a result, elasticity problems can be classed as one of four basic types [2]:

(1) <u>Elasticity Problems having both physical and geometrical linearity</u>.

In problems of this type, the effect of the angles of rotation are of the same order of magnitude as the elongations and shear strains, while the elongations do not exceed the limit of proportionality of the material. An example of this type of problem is the simple tension test when the stresses are maintained below the proportional limit.

(2) <u>Elasticity Problems which are physically non linear but geometrically linear</u>.

For these problems, angles of rotation can be neglected in projecting the forces which act on a volume element and in determining strains. However, the elongations exceed the proportional limit and require a non-linear stress-strain relation.

The simple tension test becomes a problem of this type when the stresses in the rod exceed the proportional limit.

(3) <u>Elasticity Problems linear physically, but non-linear geometrically</u>.

In problems of this type, the angles of rotation are essentially large but the strains do not exceed the limit of proportionality. An example of this type is the bending of a thin steel strip, or the buckling of a slender column within the elastic range where the original shape and position is regained after removal of the load, i.e. no permanent deformation.

(4) <u>Elasticity Problems non linear both physically and geometrically</u>.

In problems of the fourth type, the strains exceed the limit of proportionality and the angles of rotation are so large that it is necessary to retain non-linear terms both in the stress-strain relation, and in the equations of equilibrium. Plate and shell problems become examples of this type if the deformation is large and the stresses exceed the limit of proportionality.

## DISCUSSION OF PLATE PROBLEM

A plate may be defined as a three-dimensional deformable body whose thickness is small compared to its least lateral dimension. Because the thickness is small compared to the other linear dimensions, classical elasticity theory treats a relatively thin plate in an essentially two dimensional manner.

The linear small-deflection theory of plates, developed by Lagrange (1811) is based on the following assumptions [9]:

1. points which lie on a normal to the mid-plane of the undeflected plate lie on a normal to the mid-plane of the deflected plate ( $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$ ) ;

2. the stresses normal to the mid-plane of the plate, arising from the applied loading, are negligible in comparison with the stresses in the plane of the plate ( $\sigma_z << \sigma_x , \sigma_y$ ) ;

3. the slope of the deflected plate in any direction is small so that its square may be neglected in comparison with unity (curvatures $\frac{1}{r_x} = -\frac{\partial^2 w}{\partial x^2}$ , $\frac{1}{r_y} = -\frac{\partial^2 w}{\partial y^2}$ ) ;

4. the mid-plane of the plate is a neutral plane: i.e. any mid-plane stresses or membrane stresses arising from the deflection of the plate into a non-developable surface may be ignored.

In classical large deflection theory where deflection is of the order of magnitude of the plate thickness, and greater, the first three assumptions above for small deflection theory are

retained. However, the fourth assumption is no longer valid with large deflection, and middle surface stresses must be considered.

As a result of the large deflection, the problem becomes non linear. The question is, does the problem become one of geometric non linearity as assumed in classical theory, or is it one of both physical and geometrical non-linearity as proposed by this thesis? Physical non linearity is currently introduced in the inelastic buckling problem as a variation in Poisson's ratio and variation of the Elastic Modulus in the so-called tangent/secant modulus theory. Gerard, [10] provides a good "state of the art" summary of this theory. However, it is highly possible that the correct approach lies in the determination of the additional elastic constants introduced by finite deformation theory.

Assuming, for the moment, that the large deflection plate problem is physically linear, but geometrically non linear, there is some error introduced by the classical differential equations formulated by vonKármán [11] due to his treatment of the problem in a two dimensional manner. This error is negligible for very thin plates but becomes significant for plates of finite thickness. To illustrate this error, the classical large deflection assumptions will be applied to the generalized large deflection strain tensor (appendix A).

The assumption that points which lie on a normal to the mid-plane of the undeflected plate lie on a normal to the mid-plane of the deflected plate require that $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$ .

Thus the large deflection strain tensor derived in appendix A reduces to:

$$\eta = \begin{pmatrix} e_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & e_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & e_z \end{pmatrix} \quad \text{or}$$

$$(1) \quad \eta = \begin{pmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] & \frac{1}{2}\left\{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\right\} & \frac{1}{2}\left[\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right] \\ \frac{1}{2}\left\{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\right\} & \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] & \frac{1}{2}\left[\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right] \\ \frac{1}{2}\left[\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right] & \frac{1}{2}\left[\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right] & \frac{\partial w}{\partial z} + \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2 \end{pmatrix}$$

In the von Karman formulation;

$$\left(\frac{\partial u}{\partial x}\right)^2, \left(\frac{\partial u}{\partial y}\right)^2, \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}, \left(\frac{\partial v}{\partial x}\right)^2 \text{ and } \left(\frac{\partial v}{\partial y}\right)^2$$

are neglected, which seems reasonable, but even without the above terms, there are terms remaining in the expressions for $\gamma_{xz}, \gamma_{yz}$ and $e_z$ which are still significant.

If we assume that the elements of the generalized strain tensor, as applied to a plate of finite thickness, are of the order of magnitude shown below, i.e.

(1) Terms $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are of the order $\epsilon$ and are of the same order of magnitude as the derivatives of the shear forces $Q_x$ and $Q_y$.

(2) Terms $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial w}{\partial z}$, $\left(\frac{\partial w}{\partial x}\right)^2$, $\left(\frac{\partial w}{\partial y}\right)^2$ and $\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$ are of the order of magnitude $\epsilon^2$.

(3) Terms $\frac{\partial u}{\partial z}$, $\frac{\partial v}{\partial z}$, $\left(\frac{\partial u}{\partial x}\right)^2$, $\left(\frac{\partial u}{\partial y}\right)^2$, $\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}$, $\left(\frac{\partial v}{\partial y}\right)^2$ $\left(\frac{\partial v}{\partial x}\right)^2$ and $\left(\frac{\partial w}{\partial z}\right)^2$ are of the order of magnitude $\epsilon^4$ and can be neglected.

Retaining terms up to the order of magnitude $\epsilon^2$ the generalized large deflection plate strain tensor can be reduced to

$$(2)\quad \eta = \begin{pmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) & \frac{1}{2}\frac{\partial w}{\partial x} \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) & \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 & \frac{1}{2}\frac{\partial w}{\partial y} \\ \frac{1}{2}\frac{\partial w}{\partial x} & \frac{1}{2}\frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$$

If the Poisson's effect across the thickness of the plate is ignored, and $\frac{\partial w}{\partial z}$ is taken to be zero, the above strain tensor becomes identical to that obtained by Borg, Hoppe, and Kopchinski [4].

In the differential equations formulated by von Kármán [11], the strain elements are assumed to be

$$(3)\quad e_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$(4)\quad e_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$(5)\quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

By comparing the von Kármán elements with the elements of the three dimensional plate strain tensor, it is shown that von Kármán retains terms of the order $\epsilon^2$ and yet neglects terms of the order $\epsilon$ which are introduced by the shear strain elements $\gamma_{xz}$ and $\gamma_{yz}$, and also ignores the Poisson's effect across the thickness of the plate $\frac{\partial w}{\partial z}$. Thus, even with the assumption that the problem is linear physically and non-linear geometrically, a feeling for the magnitude of the error resulting from the two-dimensional approach can be achieved. This points out a significant value of the continuum mechanics approach to the problem.

At this point, it should be noted that the large deflection strain tensor was developed in terms of a Lagrangian curvilinear coordinate system which at all points remains parallel to the deformed plate (see appendix A). The treatment of a developable plate problem in the Lagrangian coordinate system lends itself readily to inextensional bending theory, but becomes a major difficulty in dealing with a flat plate with extensional plate theory. Thus, the use of the plate strain tensor derived above assumes that the deflections are small enough that an Eulerian coordinate system can be used. Even though it is recognized that error is introduced by the assumption of an Eulerian coordinate system, the plate strain tensor does provide an improvement over the von Kármán two dimensional approach as discussed above, and should lead to less error for any practical plate problem. Indeed, if point strains are sufficiently large to invalidate the above assumptions, it is probable that plastic deformation would also invalidate

the assumption of an elastic body and a non-isotropic or plastic theory would then become necessary.

In reference [12] Bleich gives an excellent historical sketch of the evolution of the von Kármán equations and subsequent strain energy approaches to the large deflection plate problem by Timoshenko, Marguerre and Trefftz. In a series of papers by Levy [13], [14], [15], [16] and [17], solutions of a theoretically exact nature were given to the von Kármán differential equation [11]. In Levy's solutions, deflections and normal pressure were expressed in the form of Fourier series and solutions were obtained for various cases of loading and support. Although the method is involved and laborious, it has been recognized as one of the more accurate solution techniques available under the von Kármán theory. However, the only plate solution known to the author using finite deformation theory incorporating the associated third-order elastic constants is a preliminary or exploratory study of a circular plate by Borg in reference [7]. Although the solution in reference [7] is close to Timoshenko's solution of a similar problem contained in reference [18], the approach to the two solutions was vastly different. Rather than using the classical approach of Timoshenko, Borg's solution is obtained from the non linear stress-strain relation developed in appendix B. This would tend to lend evidence to the validity of the Finite Deformation Theory approach to large deflection plate problems, particularly where both physical and geometrical non linearity are required.

Another non linear large deflection thin plate approach using the non linear inextensional bending theory is described by Borg in references [5] and [8]. In this approach, a theory analogous to the Murnaghan non linear elasticity theory is developed. In the non linear thin plate inextensional bending theory, it is assumed, essentially, that the large deflection is one which occurs without the development of membrane stresses and a large deflection form of the bending deformation relation is obtained. [8]. The non linearity occurs in the second derivative of the deflection terms and arises as a consequence of the analogous behavior of a thin plate to that of an elastic body in general, and in a sense, bears the same relation to linear thin plate theory that the elastica (column) solution bears to the Euler column theory.

If the deflection of a plate is not small, the assumption regarding the inextensibility of the middle surface of the plate holds only if the deflection surface is a developable surface, therefore, there is some question about the generality of the inextensional theory. Although the non linear moment-curvature tensor equations analogous to the non linear stress-strain relations, as set down by Borg [8], provides a possible solution to the non linear plate problem, it would seem that the most general differential equations would result from the non linear stress-strain relation, where the classical equations of equilibrium incorporate non linear stresses, and the Compatibility equation results from the non linear stress-strain tensor equation. In

other words, the large deflection plate problem should be treated as one which is non linear both physically and geometrically, and incorporates the third order elastic constants. Borg [6] has shown that the order of magnitude of the three additional elastic constants resulting from Murnaghan's stress-strain relation are of the same order of magnitude as the elastic constants G & E (see Table 1, page 5). As a result, their inclusion in the large deflection plate problem may be significant and would certainly warrant the derivation of more general differential equations.

DERIVATION OF PLATE COMPATIBILITY EQUATIONS

1. General Discussion

Borg, Hoppe and Kopchinski derived non linear strain compatibility equations in reference [4] by:

a) First, assuming $\sigma_z = 0$

b) Second, neglecting entirely the requirements on $\sigma_z$ and simply utilized the non linear relations for $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ as obtained from the non linear stress-strain relation.

$$(6) \quad T = \lambda I_1 E_3 + 2G\eta + (\ell I_1^2 - 2mI_2)E_3 + 2mI_1\eta + n\,co\eta$$

The major limitation of the derivation in reference [4] is that it uses a stress-strain relation which neglects compressibility terms and other terms of the same order of magnitude as those retained. The more complete stress-strain relation, as derived by Murnaghan (see appendix B) is:

$$(7) \quad T = \lambda I_1 E_3 + 2G\eta + [(\ell-\lambda) I_1^2 - 2mI_2]E_3 + 2(m+\lambda-G)I_1\eta + n\,co\eta + 4G\eta\eta$$

In addition, the derivation given in reference [4] assumes $\frac{\partial W}{\partial z} = 0$ as discussed on pages 13 and 24 which in effect says that the deformation distribution across the thickness of the plate must be constant and ignores any Poisson's effect across the thickness. This is not an unreasonable assumption for very thin plates,

but is a constraint that is open to some question for plates of finite thickness. (See the discussion of the order of magnitude of $\frac{\partial w}{\partial z}$ on page 24 following equation (19) ).

Using an approach similar to that used in reference [4], Borg derived improved compatibility equations, the results of which are reported in reference [6]. In the improved derivation Borg used the second form of the stress-strain relation that includes compressibility terms of the correct order of magnitude. Borg's derivation reported in reference [6] is verified in the derivation shown later in pages 31 to 37 . It should be noted, however, that Borg's improved derivation assumes $\frac{\partial w}{\partial z} = 0$ and thus ignores any Poisson's effect across the thickness of the plate.

Intuitive reasoning would indicate that although the large deflection plate problem is neither truly plane stress (as implied by the von Kármán theory), nor plane strain, with the exception of the very long narrow laterally loaded plate, it more closely approaches a condition of plane stress than plane strain. In linearized theory, a plane stress assumption would require that $\gamma_{xz} = \gamma_{yz} = \sigma_z \approx 0$ and $e_z = -\frac{\nu}{1-\nu}\left(e_x + e_y\right) \neq 0$

Thus it would appear that a more valid derivation in the general case, for plates of finite thickness, would be based on $\frac{\partial w}{\partial z} \neq 0$ as shown in the plate strain tensor derived on pages 11 to 13.

2. Derivation based on generalized plate strain tensor with $\frac{\partial w}{\partial z} \neq 0$

By setting $\sigma_z = 0$ and solving for $\frac{\partial w}{\partial z}$ in terms of w, x, y, u, and v, we can obtain a relationship which is in accordance with classical large deflection theory and which will be valid in the majority of cases (one exception might be relatively thick shell pressure vessels subject to very high external hydrostatic loads, i.e. very deep diving submarine pressure hulls).

Thus, solving for the terms of the stress-strain tensor equation

$$(7)\quad T = \lambda I_1 E_3 + 2G\eta + \left[(l-\lambda) I_1^2 - 2m I_2\right] E_3 + 2(m+\lambda-G) I_1 \eta + n\, co\eta + 4G\eta\eta$$

we have (retaining terms up to the order $\epsilon^2$ )

$$\eta = \begin{pmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) & \frac{1}{2}\frac{\partial w}{\partial x} \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) & \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 & \frac{1}{2}\frac{\partial w}{\partial y} \\ \frac{1}{2}\frac{\partial w}{\partial x} & \frac{1}{2}\frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(8) \qquad I_1 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$I_2 = \begin{vmatrix} \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 & \frac{1}{2}\frac{\partial w}{\partial y} \\ \frac{1}{2}\frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} + \begin{vmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 & \frac{1}{2}\frac{\partial w}{\partial x} \\ \frac{1}{2}\frac{\partial w}{\partial x} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$+ \begin{vmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) & \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \end{vmatrix}$$

or, retaining terms to the order $\epsilon^2$,

$$(9) \qquad I_2 = -\frac{1}{4}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

$Co\eta$ = Cofactor matrix of the strain tensor

$$Co\eta = \begin{pmatrix} +\begin{vmatrix} \eta_{22} & \eta_{23} \\ \eta_{32} & \eta_{33} \end{vmatrix} & -\begin{vmatrix} \eta_{21} & \eta_{23} \\ \eta_{31} & \eta_{33} \end{vmatrix} & +\begin{vmatrix} \eta_{21} & \eta_{22} \\ \eta_{31} & \eta_{33} \end{vmatrix} \\ -\begin{vmatrix} \eta_{12} & \eta_{13} \\ \eta_{32} & \eta_{33} \end{vmatrix} & +\begin{vmatrix} \eta_{11} & \eta_{13} \\ \eta_{31} & \eta_{33} \end{vmatrix} & -\begin{vmatrix} \eta_{11} & \eta_{12} \\ \eta_{31} & \eta_{32} \end{vmatrix} \\ +\begin{vmatrix} \eta_{12} & \eta_{13} \\ \eta_{22} & \eta_{23} \end{vmatrix} & -\begin{vmatrix} \eta_{11} & \eta_{13} \\ \eta_{21} & \eta_{23} \end{vmatrix} & +\begin{vmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{vmatrix} \end{pmatrix}$$

Retaining terms to the order of magnitude $\epsilon^2$, we have:

$$(10) \qquad C_0\eta = \frac{1}{4}\begin{pmatrix} -\left(\frac{\partial w}{\partial y}\right)^2 & \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & 0 \\ \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & -\left(\frac{\partial w}{\partial x}\right)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(11) \qquad \eta\eta = \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix}\begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \left(\frac{\partial w}{\partial x}\right)^2 & \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & 0 \\ \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & \left(\frac{\partial w}{\partial y}\right)^2 & 0 \\ 0 & 0 & \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \end{pmatrix}$$

$I_1^2$ and $I_1\eta$ contain terms of order higher than $\epsilon^2$ and are neglected.

Thus, writing out the stress-strain tensor equation, we have:

$$(12) \qquad T = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} = \lambda\left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]\right\}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$+ 2G\begin{pmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) & \frac{1}{2}\frac{\partial w}{\partial x} \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) & \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 & \frac{1}{2}\frac{\partial w}{\partial y} \\ \frac{1}{2}\frac{\partial w}{\partial x} & \frac{1}{2}\frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} + \frac{m}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$+ \frac{n}{4}\begin{pmatrix} -\left(\frac{\partial w}{\partial y}\right)^2 & \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & 0 \\ \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & -\left(\frac{\partial w}{\partial y}\right)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + G\begin{pmatrix} \left(\frac{\partial w}{\partial x}\right)^2 & \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & 0 \\ \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & \left(\frac{\partial w}{\partial y}\right)^2 & 0 \\ 0 & 0 & \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \end{pmatrix}$$

The equations for the elements of the stress tensor can now be written

(13) $$\sigma_x = \lambda\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+\frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\right\}+2G\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]$$
$$+\frac{m}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]-\frac{n}{4}\left(\frac{\partial w}{\partial y}\right)^2+G\left(\frac{\partial w}{\partial x}\right)^2$$

(14) $$\sigma_y = \lambda\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+\frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\right\}+2G\left[\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]$$
$$+\frac{m}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]-\frac{n}{4}\left(\frac{\partial w}{\partial x}\right)^2+G\left(\frac{\partial w}{\partial y}\right)^2$$

(15) $$\sigma_z = \lambda\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+\frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\right\}+2G\frac{\partial w}{\partial z}$$
$$+\frac{m}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]+G\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]$$

(16) $$\tau_{xy} = \tau_{yx} = G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\left(\frac{n}{4}+2G\right)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

(17) $$\tau_{xz} = \tau_{zx} = G\frac{\partial w}{\partial x}$$

(18) $$\tau_{yz} = \tau_{zy} = G\frac{\partial w}{\partial y}$$

Assuming $\sigma_z = 0$ as stated before, we have

$$\sigma_z = 0 = (\lambda+2G)\frac{\partial w}{\partial z} + \lambda\left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]\right\} + \left(\frac{m}{2} + G\right)\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

solving for $\frac{\partial w}{\partial z}$

(19) $$\frac{\partial w}{\partial z} = -\left(\frac{\lambda}{\lambda+2G}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{(\lambda+m+2G)}{2(\lambda+2G)}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

It should be noted that $\frac{\lambda}{\lambda+2G} = \frac{\nu}{1-\nu}$ which for steel where $\nu = .3$ is about .43.

Thus $\frac{\partial w}{\partial z}$ contains terms of the same order of magnitude as the terms retained in the basic derivation, ( $\epsilon^2$ ), and is of the same order of magnitude as the remaining terms in the first invariant of the strain tensor, and should, therefore, not be neglected.

Substituting the expression for $\frac{\partial w}{\partial z}$ into the expression for $\sigma_x$ and $\sigma_y$, we have

(20a) $$\sigma_x = \lambda\left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \left(\frac{\lambda}{\lambda+2G}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{(\lambda+m+2G)}{2(\lambda+2G)}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]\right\}$$
$$+ 2G\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + \frac{m}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] - \frac{n}{4}\left(\frac{\partial w}{\partial y}\right)^2 + G\left(\frac{\partial w}{\partial x}\right)^2$$

(21a) $$\sigma_y = \lambda\left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \left(\frac{\lambda}{\lambda+2G}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{(\lambda+m+2G)}{2(\lambda+2G)}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]\right\}$$
$$+ 2G\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{m}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] - \frac{n}{4}\left(\frac{\partial w}{\partial x}\right)^2 + G\left(\frac{\partial w}{\partial y}\right)^2$$

Rearranging terms, we have

(20b) $$\sigma_x = 2G\left(\frac{\lambda}{\lambda+2G}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left[\frac{\lambda}{2}\left(\frac{\lambda}{\lambda+2G}\right)+\frac{m+2G}{2}\left(\frac{\lambda}{\lambda+2G}\right)-\frac{\lambda}{2}-\frac{m}{2}\right]\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]$$
$$+2G\frac{\partial u}{\partial x}-\frac{n}{4}\left(\frac{\partial w}{\partial y}\right)^2+2G\left(\frac{\partial w}{\partial x}\right)^2$$

(21b) $$\sigma_y = 2G\left(\frac{\lambda}{\lambda+2G}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left[\frac{\lambda}{2}\left(\frac{\lambda}{\lambda+2G}\right)+\frac{m+2G}{2}\left(\frac{\lambda}{\lambda+2G}\right)-\frac{\lambda}{2}-\frac{m}{2}\right]\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]$$
$$+2G\frac{\partial v}{\partial y}-\frac{n}{4}\left(\frac{\partial w}{\partial x}\right)^2+2G\left(\frac{\partial w}{\partial y}\right)^2$$

since $\frac{\lambda}{\lambda+2G} = \frac{\nu}{1-\nu}$ and $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = \frac{2G\nu}{1-2\nu}$

we can write the above expressions for $\sigma_x$ and $\sigma_y$ in terms of the engineering constants

(20c) $$\sigma_x = \left(\frac{2G\nu}{1-\nu}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left[-\frac{m}{2}+\frac{G\nu}{1-2\nu}\left(\frac{\nu}{1-\nu}\right)+\left(\frac{m+2G}{2}\right)\left(\frac{\nu}{1-\nu}\right)-\left(\frac{G\nu}{1-2\nu}\right)\right]\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]$$
$$+2G\left(\frac{\partial u}{\partial x}\right)-\frac{n}{4}\left(\frac{\partial w}{\partial y}\right)^2+2G\left(\frac{\partial w}{\partial x}\right)^2$$

(21c) $$\sigma_y = \left(\frac{2G\nu}{1-\nu}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left[-\frac{m}{2}+\frac{G\nu}{1-2\nu}\left(\frac{\nu}{1-\nu}\right)+\left(\frac{m+2G}{2}\right)\left(\frac{\nu}{1-\nu}\right)-\left(\frac{G\nu}{1-2\nu}\right)\right]\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]$$
$$+2G\left(\frac{\partial v}{\partial y}\right)-\frac{n}{4}\left(\frac{\partial w}{\partial x}\right)^2+2G\left(\frac{\partial w}{\partial y}\right)^2$$

If we now define coefficients as

(22) $$A = \frac{2G\nu}{1-\nu}$$

(23) $$B = -\left[-\frac{m}{2} + \left(\frac{G\nu}{1-2\nu}\right)\left(\frac{\nu}{1-\nu}\right) + \left(\frac{m+2G}{2}\right)\left(\frac{\nu}{1-\nu}\right) - \left(\frac{G\nu}{1-2\nu}\right)\right]$$

$$B = -\left[-\frac{m}{2}\frac{(1-2\nu)}{1-\nu} + \frac{G\nu}{1-\nu} - \frac{G\nu}{1-\nu}\right]$$

$$B = +\frac{m}{2}\left(\frac{1-2\nu}{1-\nu}\right)$$

(24) $$C = 2G$$

(25) $$D = -\frac{n}{4}$$

we can write

(26) $$\sigma_x = A\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + B\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + C\frac{\partial u}{\partial x} + D\left(\frac{\partial w}{\partial y}\right)^2 + C\left(\frac{\partial w}{\partial x}\right)^2$$

(27) $$\sigma_y = A\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + B\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + C\frac{\partial v}{\partial y} + D\left(\frac{\partial w}{\partial x}\right)^2 + C\left(\frac{\partial w}{\partial y}\right)^2$$

(28) $$\tau_{xy} = \frac{C}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + (C-D)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

Since we want to ultimately arrive at an equation in the form

$\frac{\partial^2 \sigma_x}{\partial y^2} - 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_y}{\partial x^2}$ which satisfies the stress function relation $\nabla^4 F = \frac{\partial^4 F}{\partial y^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial x^4}$

where $F$ is defined by $\sigma_x = \frac{\partial^2 F}{\partial y^2}$ ; $\sigma_y = \frac{\partial^2 F}{\partial x^2}$ ; $\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$

as developed in the classical theory, we perform the indicated differentiation.

$$(29) \quad \frac{\partial^2 \sigma_x}{\partial y^2} = A \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + B \frac{\partial^2}{\partial y^2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + C \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right) + D \frac{\partial^2}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2 + C \frac{\partial^2}{\partial y^2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$(30) \quad \frac{\partial^2 \sigma_y}{\partial x^2} = A \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + B \frac{\partial^2}{\partial x^2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + C \frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial y}\right) + D \frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + C \frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$(31) \quad 2 \frac{\partial^2 \tau_{xy}}{\partial x \partial y} = C \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 2(C - D) \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y}\right)$$

Thus

$$(32a) \quad \nabla^4 F = \frac{\partial^2 \sigma_x}{\partial y^2} - 2 \frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_y}{\partial x^2} = A \nabla^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + B \mathcal{J}_{ILD}$$

$$+ C\left[\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial y}\right) - \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]$$

$$+ C\left[\frac{\partial^2}{\partial y^2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)^2\right] + D\left[\frac{\partial^2}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$+ 2(D - C) \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y}\right)$$

where $\mathcal{J}_{ILD} = \nabla^2\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$ = 1st invariant of the large deflection tensor.

From the identity $\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

the third term goes to zero and we are left with

$$(32b) \quad \nabla^4 F = A\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + B\mathcal{J}_{ILD} + C\left[\frac{\partial^2}{\partial y^2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)^2 - 2\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\right]$$

$$+ D\left[\frac{\partial^2}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + 2\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\right]$$

Carrying out the differentiation in the last two terms, we have

$$C\left[2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 + 2\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x\partial y^2} + 2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 + 2\frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x^2\partial y} - 2\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x\partial y^2} - 2\frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x^2\partial y} - 2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]$$

$$= -2C\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]$$

and $$2D\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3} + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial y^3} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x^2\partial y} + \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]$$

If we consider the second invariant of the curvature tensor

$$\mathcal{J}_{2c} = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2$$

and considering the expanded form of the 1st invariant of the large deflection tensor $\mathcal{J}_{ILD} = \nabla^2\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = 2\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right.$

$$\left. + \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3} + \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial y^3} + \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x\partial y^2} + \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x^2\partial y}\right]$$

By adding and subtracting $2D\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2$

we can write the last two terms as $-2C\mathcal{J}_{2c} + D\mathcal{J}_{ILD} + 2D\mathcal{J}_{2c}$

Thus, the equation for $\nabla^4 F$ can be written as:

$$(33) \quad \nabla^4 F = A\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + (B+D)\,\mathcal{J}_{1LD} + 2(D-C)\,\mathcal{J}_{2C}$$

In order to get rid of the term $A\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$ we go back to the stress relations

$$(26) \quad \sigma_x = A\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + B\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] + C\frac{\partial u}{\partial x} + D\left(\frac{\partial w}{\partial y}\right)^2 + C\left(\frac{\partial w}{\partial x}\right)^2$$

$$(27) \quad \sigma_y = A\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + B\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] + C\frac{\partial v}{\partial y} + D\left(\frac{\partial w}{\partial x}\right)^2 + C\left(\frac{\partial w}{\partial y}\right)^2$$

Adding we have

$$(34) \quad \sigma_x+\sigma_y = 2A\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + 2B\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] + C\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + (D+C)\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]$$

solving for $\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$ we have

$$(35) \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = \left(\frac{1}{2A+C}\right)\left\{\sigma_x+\sigma_y - (2B+D+C)\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\right\}$$

Now taking the Laplacian of the above expression, we have

$$(36) \quad \nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = \left(\frac{1}{2A+C}\right)\left[\nabla^2(\sigma_x+\sigma_y) - (2B+D-C)\,\mathcal{J}_{1LD}\right]$$

Taking the Laplacian $\nabla^2(\sigma_x+\sigma_y)$, by adding and subtracting

$2\frac{\partial^2\tau_{xy}}{\partial x\partial y}$ we have $\left(\frac{\partial^2\sigma_x}{\partial y^2} - 2\frac{\partial^2\tau_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_y}{\partial x^2}\right) + \left(\frac{\partial^2\sigma_x}{\partial x^2} + 2\frac{\partial^2\tau_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_y}{\partial y^2}\right)$

$$= \left(\frac{\partial^4 F}{\partial y^4} + 2\frac{\partial^4 F}{\partial x^2\partial y^2} + \frac{\partial^4 F}{\partial x^4}\right) + \left(\frac{\partial^4 F}{\partial x^2\partial y^2} - 2\frac{\partial^4 F}{\partial x^2\partial y^2} + \frac{\partial^4 F}{\partial x^2\partial y^2}\right)$$

$$= \nabla^4 F$$

Thus we have

(37) $$\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\left(\frac{1}{2A+C}\right)\left\{\nabla^4 F-(2B+D+C)\mathcal{J}_{1LD}\right\}$$

Substituting this expression into

(33) $$\nabla^4 F = A\left[\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]+(B+D)\mathcal{J}_{1LD}+2(D-C)\mathcal{J}_{2C}$$

gives

(34) $$\nabla^4 F=\left(\frac{2A+C}{A+C}\right)\left\{\left[B+D-\frac{A(2B+D+C)}{2A+C}\right]\mathcal{J}_{1LD}+2(D-C)\mathcal{J}_{2C}\right\}$$

since $A\equiv\frac{2G\nu}{1-\nu}$ ; $B\equiv\frac{m}{2}\frac{(1-2\nu)}{(1-\nu)}$ ; $C=2G$ ; $D=-\frac{n}{4}$

we can write the above equation in terms of the elastic constants.

Since $\left[(B+D)-\frac{A(2B+D+C)}{2A+C}\right]=\left(\frac{1}{\nu+1}\right)\left[\frac{m}{2}(1-2\nu)-\frac{n}{4}-2G\nu\right]$ ,

$$\frac{2A+C}{A+C}=\nu+1$$

and $2(D-C)=-2\left(\frac{n}{4}+2G\right)=-\left(4G+\frac{n}{2}\right)$

we have finally

(35) $$\nabla^4 F=\left[\frac{m}{2}(1-2\nu)-\frac{n}{4}-2G\nu\right]\mathcal{J}_{1LD}-(\nu+1)\left(4G+\frac{n}{2}\right)\mathcal{J}_{2C}$$

3. Derivation of Plate Compatibility Equations based on the assumption $\frac{\partial w}{\partial z} = 0$

For the assumption $\frac{\partial w}{\partial z} = 0$, we have the identical expression for the strain tensor as derived by Borg, Hoppe and Kopchinski [4];

$$(36)\quad \eta = \begin{pmatrix} e_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & e_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & e_z \end{pmatrix} =$$

$$\eta = \begin{pmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) & \frac{1}{2}\frac{\partial w}{\partial x} \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) & \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 & \frac{1}{2}\frac{\partial w}{\partial y} \\ \frac{1}{2}\frac{\partial w}{\partial x} & \frac{1}{2}\frac{\partial w}{\partial y} & 0 \end{pmatrix}$$

Thus, the terms of the stress-strain tensor equation

$$(37)\quad T = \lambda I_1 E_3 + 2G\eta + \left[(l-\lambda) I_1^2 - 2m I_2\right] E_3 + 2(m+\lambda-G) I_1 \eta + n\,\mathrm{Co}\,\eta + 4G\eta\eta$$

are (retaining terms to the order $\epsilon^2$):

$$(38)\quad I_1 = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$(39)\quad I_2 = -\frac{1}{4}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$(40) \qquad Co\eta = \frac{1}{4}\begin{pmatrix} -\left(\frac{\partial w}{\partial y}\right)^2 & \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & 0 \\ \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & -\left(\frac{\partial w}{\partial x}\right)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(41) \qquad \eta\eta = \frac{1}{4}\begin{pmatrix} \left(\frac{\partial w}{\partial x}\right)^2 & \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & 0 \\ \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & \left(\frac{\partial w}{\partial y}\right)^2 & 0 \\ 0 & 0 & \left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] \end{pmatrix}$$

Terms $I_1\eta$ and $I_1^2$ contain terms of the order $\epsilon^4$ and can be neglected.

Substituting the above expressions into the stress-strain relation gives

$$(42) \qquad T = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} = \lambda\left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\right\}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$+2G\begin{pmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) & \frac{1}{2}\frac{\partial w}{\partial x} \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) & \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 & \frac{1}{2}\frac{\partial w}{\partial y} \\ \frac{1}{2}\left(\frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y}\right) & 0 \end{pmatrix}$$

$$+\frac{m}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{n}{4}\begin{pmatrix} -\left(\frac{\partial w}{\partial y}\right)^2 & \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & 0 \\ \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & -\left(\frac{\partial w}{\partial x}\right)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+G\begin{pmatrix} \left(\frac{\partial w}{\partial x}\right)^2 & \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & 0 \\ \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} & \left(\frac{\partial w}{\partial y}\right)^2 & 0 \\ 0 & 0 & \left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] \end{pmatrix}$$

The equations for the elements of the stress tensor (equation [42] ) are as follows:

$$(43) \quad \sigma_x = \lambda\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\right\} + 2G\left[\frac{\partial u}{\partial x}+\left(\frac{\partial w}{\partial x}\right)^2\right] + \frac{m}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] - \frac{n}{4}\left(\frac{\partial w}{\partial y}\right)^2$$

$$(44) \quad \sigma_y = \lambda\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\right\} + 2G\left[\frac{\partial v}{\partial y}+\left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{m}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] - \frac{n}{4}\left(\frac{\partial w}{\partial x}\right)^2$$

$$(45) \quad \sigma_z = \lambda\left\{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]\right\} + \frac{m}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] + G\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$(46) \quad \tau_{xy} = \tau_{yx} = G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + \left(\frac{n}{4}+2G\right)\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

$$(47) \quad \tau_{xz} = \tau_{zx} = G\,\frac{\partial w}{\partial x}$$

$$(48) \quad \tau_{yz} = \tau_{zy} = G\,\frac{\partial w}{\partial y}$$

Since, regardless of assumptions, we want to eventually have an equation in terms of a stress function $F$ or $\nabla^4 F$ where:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} \qquad \text{or} \qquad \frac{\partial^2 \sigma_x}{\partial y^2} = \frac{\partial^4 F}{\partial y^4}$$

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} \qquad \text{or} \qquad \frac{\partial^2 \sigma_y}{\partial x^2} = \frac{\partial^4 F}{\partial x^4}$$

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \qquad \text{or} \qquad -2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 2\frac{\partial^4 F}{\partial x^2 \partial y^2}$$

we can carry out the indicated differentiation and add the resulting expressions

Thus, we have

$$(49)\quad \frac{\partial^2\sigma_x}{\partial y^2} = \lambda\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + \frac{\lambda}{2}\frac{\partial^2}{\partial y^2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] + 2G\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right)$$

$$+2G\frac{\partial^2}{\partial y^2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{m}{2}\frac{\partial^2}{\partial y^2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] - \frac{n}{4}\frac{\partial^2}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$(50)\quad \frac{\partial^2\sigma_y}{\partial x^2} = \lambda\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + \frac{\lambda}{2}\frac{\partial^2}{\partial x^2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] + 2G\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial y}\right)$$

$$+2G\frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{m}{2}\frac{\partial^2}{\partial x^2}\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] - \frac{n}{4}\frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$(51)\quad 2\frac{\partial^2\tau_{xy}}{\partial x\,\partial y} = 2G\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + 2\left(\frac{n}{4}+2G\right)\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

Thus

$$(52a)\quad \nabla^4 F = \frac{\partial^2\sigma_x}{\partial y^2} - 2\frac{\partial^2\tau_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_y}{\partial x^2} = \lambda\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + \left(\frac{\lambda+m}{2}\right)\nabla^2\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$+2G\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right)-\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]$$

$$+2G\left[\frac{\partial^2}{\partial y^2}\left(\frac{\partial w}{\partial x}\right)^2+\frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial y}\right)^2-2\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\right]$$

$$-\frac{n}{4}\left[\frac{\partial^2}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{\partial^2}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2+2\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\right]$$

From the identity $\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

and the definition of the 1st invariant of the large deflection Tensor $\mathcal{J}_{ILD} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = \nabla^2\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$

we have (carrying out the remainder of the differentiation)

(52b) $$\nabla^4 F = \lambda \nabla^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\lambda + m}{2}\mathcal{J}_{ILD} - 4G\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]$$
$$-\frac{n}{4}\left\{2\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^3} + \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial y^3} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right.\right.$$
$$\left.\left. + \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial w}{\partial y}\frac{\partial^3 w}{\partial x^2 \partial y} + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\}$$

From the second invariant of the thin plate curvature tensor

$$\mathcal{J}_{2C} = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$$

by adding and subtracting $-\frac{n}{2}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$ we have

(52c) $$\nabla^4 F = \lambda \nabla^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left[\frac{\lambda + m}{2}\right]\mathcal{J}_{ILD} - 4G\mathcal{J}_{2C} - \frac{n}{4}\left(\mathcal{J}_{ILD} + 2\mathcal{J}_{2C}\right)$$

Combining terms we have

(53) $$\nabla^4 F = \lambda \nabla^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{[2(\lambda + m) - n]}{4}\mathcal{J}_{ILD} - \left(4G + \frac{n}{2}\right)\mathcal{J}_{2C}$$

If we now assume $\sigma_z = 0$, we have from equation (45)

$$\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{(\lambda + m + 2G)}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \quad \text{or}$$

$$\lambda \nabla^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{(\lambda + m + 2G)}{2}\mathcal{J}_{ILD}$$

Substituting this expression into equation (53) above, we have

$$(54)\qquad \underset{(\sigma_z=0)}{\nabla^4 F} = -\left(G+\frac{n}{4}\right)\mathcal{J}_{ILD} - \left(4G+\frac{n}{2}\right)\mathcal{J}_{2C}$$

(This form agrees with that derived by Borg [6] )

If we now make no assumption on $\sigma_z$ and work entirely with the elements of the stress tensor $\sigma_x$ , $\sigma_y$ and $\tau_{xy}$ , as was done in reference [4], we have, by adding equations (43) and (44)

$$(55)\qquad \sigma_x+\sigma_y = 2(\lambda+G)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\lambda+m+2G-\frac{n}{4}\right)\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]$$

Taking the Laplacian of each term, we have

$$(56)\qquad \nabla^2(\sigma_x+\sigma_y) = 2(\lambda+G)\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\lambda+m+2G-\frac{n}{4}\right)\mathcal{J}_{ILD}$$

By adding and substracting $2\frac{\partial^2\tau_{xy}}{\partial x\partial y}$ , we have

$$(57)\qquad \nabla^4 F = 2(\lambda+G)\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\lambda+m+2G-\frac{n}{4}\right)\mathcal{J}_{ILD}$$

Thus, solving for $\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$, we have

$$(58)\qquad \nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = \frac{\nabla^4 F}{2(\lambda+G)} - \frac{\left(\lambda+m+2G-\frac{n}{4}\right)}{2(\lambda+G)}\mathcal{J}_{ILD}$$

Substituting equation (58) into equation (53), we have

$$(59)\qquad \nabla^4 F = \frac{\lambda}{2(\lambda+G)}\nabla^4 F+\left\{-\frac{\lambda}{2(\lambda+G)}\left(\lambda+m+2G-\frac{n}{4}\right)+\frac{2(\lambda+m)-n}{4}\right\}\mathcal{J}_{ILD}-\left(4G+\frac{n}{2}\right)\mathcal{J}_{2C}$$

Combining terms and reducing, we have

$$(60)\qquad \nabla^4 F = \frac{2(\lambda+G)}{\lambda+2G}\left\{-\frac{\lambda(\lambda+m+2G-n/4)}{2(\lambda+G)}+\frac{2(\lambda+m)-n}{4}\right\}\mathcal{J}_{ILD}-\frac{2(\lambda+G)}{\lambda+2G}\left(4G+\frac{n}{2}\right)\mathcal{J}_{2C}$$

or since $\frac{\lambda}{\lambda+2G} = \frac{\nu}{1-\nu}$ ; $\lambda = \frac{2G\nu}{1-2\nu}$ ; $\lambda+G = \frac{G}{1-2\nu}$

$$2\left(\frac{\lambda+G}{\lambda+2G}\right) = \frac{1}{1-\nu} \quad ; \quad \nu = \frac{\lambda}{2(\lambda+G)}$$

we have

(61) $$\nabla^4 F = \left(\frac{1}{1-\nu}\right)\left\{-\nu\left(\frac{2G\nu}{1-2\nu} + m + 2G - \frac{n}{4}\right) + \frac{2\left(\frac{2G\nu}{1-2\nu} + m\right) - n}{4}\right\} \mathcal{J}_{1LD}$$
$$- \left(\frac{1}{1-\nu}\right)\left(4G + \frac{n}{2}\right) \mathcal{J}_{2C}$$

Reducing and collecting terms, we have

(62) $$\nabla^4 F = \left(\frac{1}{1-\nu}\right)\left[-G\nu - \frac{n}{4}(1-\nu) + \frac{m}{2}(1-2\nu)\right] \mathcal{J}_{1LD} - \left(\frac{1}{1-\nu}\right)\left(4G+\frac{n}{2}\right) \mathcal{J}_{2C}$$

( $\frac{\partial w}{\partial z} = 0$ , No restriction on $\sigma_z$ )

Equation (62) is identical to that derived by Borg [6].

Thus, we now have three new compatibility equations under the assumptions indicated that can be tailored to a particular problem.

(35) $$\nabla^4 F = \left[\frac{m}{2}(1-2\nu) - \frac{n}{4} - 2G\nu\right] \mathcal{J}_{1LD} - (\nu+1)\left(4G + \frac{n}{2}\right) \mathcal{J}_{2C}$$

$$\left(\begin{matrix} \frac{\partial w}{\partial z} \neq 0 \\ \sigma_z = 0 \end{matrix}\right)$$

FOR USE IN PROBLEMS THAT APPROACH PLANE STRESS

(54) $$\nabla^4 F = -\left(G + \frac{n}{4}\right)\mathfrak{g}_{1LD} - \left(4G + \frac{n}{2}\right)\mathfrak{g}_{2C}$$

$\left(\begin{array}{l} \dfrac{\partial w}{\partial z} = 0 \\ \sigma_z = 0 \end{array}\right)$ FOR USE IN VERY THIN PLATES WHERE POISSON'S EFFECT ACROSS THE THICKNESS CAN BE IGNORED AND WHERE THE PROBLEM APPROACHES PLANE STRESS

(62) $$\nabla^4 F = \left[\frac{m}{2}\frac{(1-2\nu)}{(1-\nu)} - \frac{n}{4} - \frac{G\nu}{1-\nu}\right]\mathfrak{g}_{1LD} - \frac{1}{(1-\nu)}\left(4G + \frac{n}{2}\right)\mathfrak{g}_{2C}$$

$\left(\begin{array}{l} \dfrac{\partial w}{\partial z} = 0 \\ \text{No restriction on } \sigma_z \end{array}\right)$ FOR USE IN PROBLEMS THAT APPROACH PLANE STRAIN OR WHERE $\sigma_z$ MAY NOT BE NEGLIGIBLE AND $\frac{\partial w}{\partial z} = 0$ IS JUSTIFIED

And finally, for comparison, the classical von Kármán equation

(63) $$\nabla^4 F = -E\,\mathfrak{g}_{2C} = -(\nu+1)\,2G\,\mathfrak{g}_{2C}$$

## DERIVATION OF EQUILIBRIUM EQUATIONS

In the classical linear theory of elasticity, the equations of equilibrium are

$$\operatorname{div} T + \rho F = 0$$

or

$$(64) \quad \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho_x F_x = 0$$

$$(65) \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho_y F_y = 0$$

$$(66) \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \rho_z F_z = 0$$

However, for the large deformation case, one must remember that the above equations are referred to the original coordinate system and that the curvilinear coordinates a, b, c, are identified with x, y, and z so that when the forces are projected, the changes in position of the points of the body due to its deformation are neglected. Stated another way, in the linear theory, no distinction is made between undeformed and deformed values of the magnitudes and positions of the elemental areas on which the stresses act. In other words, in projecting the forces, the rotation which an element of volume experiences as a result of deformation is neglected. This assumption is open to question and is far from admissible in the general case. In the general case, it is necessary to take into account the fact that differentiation is with respect to a, b, c (the coordinates of the points before deformation). Since the nature of the simplified linear equations of

equilibrium rests on the assumption of small elongations, shears and angles of rotation, and in view of the interaction of these quantities in the higher order equations, whether or not the non-linear terms can be neglected depends not only on the magnitude of the terms of the strain tensor, but also on the comparitive magnitude of the corresponding terms of the stress tensor. It follows that the smallness of the angles of rotation in comparison to unity is not a sufficient condition for linearization of the equations of equilibrium. It is also essential to know whether the stresses which are multiplied by rotations are large in comparison with those stresses which enter linearly into the equations. The problems of elastic stability or of thin plates with large deformation are cases in point.

The curvilinear form of the equations of equilibrium

$$(67) \qquad \frac{\partial \sigma_a}{\partial a} + \frac{\partial \tau_{ba}}{\partial b} + \frac{\partial \tau_{ca}}{\partial c} + \rho_a F_a = 0$$

$$(68) \qquad \frac{\partial \tau_{ab}}{\partial a} + \frac{\partial \sigma_b}{\partial b} + \frac{\partial \tau_{cb}}{\partial c} + \rho_b F_b = 0$$

$$(69) \qquad \frac{\partial \tau_{ac}}{\partial a} + \frac{\partial \tau_{bc}}{\partial b} + \frac{\partial \sigma_c}{\partial c} + \rho_c F_c = 0$$

can be transformed to the cartesian coordinates of the points of the body before its deformation. Novozhilov [2] has derived a vector form of the generalized equations of equilibrium as:

$$(70) \quad \frac{\partial}{\partial x}\left(\frac{S_x^*}{S_x}\,\sigma_{n_1}\right) + \frac{\partial}{\partial y}\left(\frac{S_y^*}{S_y}\,\sigma_{n_2}\right) + \frac{\partial}{\partial z}\left(\frac{S_z^*}{S_z}\,\sigma_{n_3}\right) + \frac{V^*}{V}\,F^* = 0$$

where $\sigma_{n_1}$, $\sigma_{n_2}$ and $\sigma_{n_3}$ are vectors,

$$\left.\begin{aligned}
\frac{S_x^*}{S_x} &= \sqrt{(1+2e_y)(1+2e_z) - \left(\tfrac{1}{2}\gamma_{yz}\right)^2} \\
\frac{S_y^*}{S_y} &= \sqrt{(1+2e_x)(1+2e_z) - \left(\tfrac{1}{2}\gamma_{xz}\right)^2} \\
\frac{S_z^*}{S_z} &= \sqrt{(1+2e_x)(1+2e_y) - \left(\tfrac{1}{2}\gamma_{xy}\right)^2}
\end{aligned}\right\}$$

Ratios of area of elemental sides before deformation to area of sides after deformation.

$\dfrac{V^*}{V}$ = ratio of volume of element after deformation to volume of element before deformation = D

$$D = \begin{vmatrix} 1+\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & 1+\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & 1+\frac{\partial w}{\partial z} \end{vmatrix} = |J|_{x,y,z} = \sqrt{1+2I_1+4I_2+8I_3}$$

and $I_1$, $I_2$ and $I_3$ = 1st, 2nd and 3rd invariants of strain tensor

Resolving the vectors $\sigma_{n_1}$, $\sigma_{n_2}$, $\sigma_{n_3}$ in the directions i, j and k, we can write Novozhilov's equation

$$(71)\quad \frac{\partial}{\partial x}\left[\frac{S_x^*}{S_x}\left(\sigma_x i + \tau_{xy} j + \tau_{xz} k\right)\right] + \frac{\partial}{\partial y}\left[\frac{S_y^*}{S_y}\left(\tau_{yx} i + \sigma_y j + \tau_{yz} k\right)\right]$$

$$+\frac{\partial}{\partial z}\left[\frac{S_z^*}{S_z}\left(\tau_{zx} i + \tau_{zy} j + \sigma_z k\right)\right] + \frac{V^*}{V} F^* = 0$$

Projecting this vector relation on the x, y, z axes we have:

$$(72)\quad \frac{\partial}{\partial x}\left[(1+e_x)\sigma_x^{*} + \left(\tfrac{1}{2}e_{xy}-\omega_z\right)\tau_{xy}^{*} + \left(\tfrac{1}{2}e_{xz}+\omega_y\right)\tau_{xz}^{*}\right]$$
$$+\frac{\partial}{\partial y}\left[(1+e_x)\tau_{yx}^{*} + \left(\tfrac{1}{2}e_{xy}-\omega_z\right)\sigma_y^{*} + \left(\tfrac{1}{2}e_{xz}+\omega_y\right)\tau_{yz}^{*}\right]$$
$$+\frac{\partial}{\partial z}\left[(1+e_x)\tau_{zx}^{*} + \left(\tfrac{1}{2}e_{xy}-\omega_z\right)\tau_{zy}^{*} + \left(\tfrac{1}{2}e_{xz}+\omega_y\right)\sigma_z^{*}\right] + \frac{V^{*}}{V}F_a = 0$$

$$(73)\quad \frac{\partial}{\partial x}\left[\left(\tfrac{1}{2}e_{xy}+\omega_z\right)\sigma_x^{*} + (1+e_y)\tau_{xy}^{*} + \left(\tfrac{1}{2}e_{yz}-\omega_x\right)\tau_{xz}^{*}\right]$$
$$+\frac{\partial}{\partial y}\left[\left(\tfrac{1}{2}e_{xy}+\omega_z\right)\tau_{yx}^{*} + (1+e_y)\sigma_y^{*} + \left(\tfrac{1}{2}e_{yz}-\omega_x\right)\tau_{yz}^{*}\right]$$
$$+\frac{\partial}{\partial z}\left[\left(\tfrac{1}{2}e_{xy}+\omega_z\right)\tau_{zx}^{*} + (1+e_y)\tau_{zy}^{*} + \left(\tfrac{1}{2}e_{yz}-\omega_x\right)\sigma_z^{*}\right] + \frac{V^{*}}{V}F_b = 0$$

$$(74)\quad \frac{\partial}{\partial x}\left[\left(\tfrac{1}{2}e_{xz}-\omega_y\right)\sigma_x^{*} + \left(\tfrac{1}{2}e_{yz}+\omega_x\right)\tau_{xy}^{*} + (1+e_z)\tau_{xz}^{*}\right]$$
$$+\frac{\partial}{\partial y}\left[\left(\tfrac{1}{2}e_{xz}-\omega_y\right)\tau_{yx}^{*} + \left(\tfrac{1}{2}e_{yz}+\omega_x\right)\sigma_y^{*} + (1+e_z)\tau_{yz}^{*}\right]$$
$$+\frac{\partial}{\partial z}\left[\left(\tfrac{1}{2}e_{xz}-\omega_y\right)\tau_{zx}^{*} + \left(\tfrac{1}{2}e_{yz}+\omega_x\right)\tau_{zy}^{*} + (1+e_z)\sigma_z^{*}\right] + \frac{V^{*}}{V}F_c = 0$$

where $e_x = \frac{\partial u}{\partial x}$ ; $e_y = \frac{\partial v}{\partial y}$ ; $e_z = \frac{\partial w}{\partial z}$ ; $e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ ;

$e_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$ ; $e_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$ ; $2\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial x}$

$2\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$ ; $2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ ; $\sigma_x^{*} = \frac{S_x^{*}}{S_x}\,\frac{\sigma_x}{1+E_x}$

$\tau_{xy}^{*} = \frac{S_x^{*}}{S_x}\,\frac{\tau_{xy}}{1+E_y}$ ; $\tau_{xz} = \frac{S_x^{*}}{S_x}\,\frac{\tau_{xz}}{1+E_z}$ ; $\sigma_y^{*} = \frac{S_y^{*}}{S_y}\,\frac{\sigma_y}{1+E_y}$ ; $\tau_{yz}^{*} = \frac{S_y^{*}}{S_y}\,\frac{\tau_{yz}}{1+E_z}$

$\tau_{yx}^{*} = \frac{S_y^{*}}{S_y}\,\frac{\tau_{yx}}{1+E_x}$ ; $\sigma_z^{*} = \frac{S_z^{*}}{S_z}\,\frac{\sigma_z}{1+E_z}$ ; $\tau_{zx}^{*} = \frac{S_z^{*}}{S_z}\,\frac{\tau_{zx}}{1+E_x}$ ; $\tau_{zy}^{*} = \frac{S_z^{*}}{S_z}\,\frac{\tau_{zy}}{1+E_y}$

$E_x$ , $E_y$ and $E_z$ = Elongations in fibers in directions x,y and z

The above values of normal and shear stresses are not, strictly speaking, stresses. They can be called stresses referred to the dimensions of an element of volume before, not after, the deformation. [2]

Writing the non linear form of the general equation of equilibrium in terms of $u$, $v$ and $w$ (the displacements) we have:

$$(75)\quad \left(1+\frac{\partial u}{\partial x}\right)\left(\frac{\partial \sigma_x^*}{\partial x}+\frac{\partial \tau_{xy}^*}{\partial y}+\frac{\partial \tau_{xz}^*}{\partial z}\right)+\frac{\partial u}{\partial y}\left(\frac{\partial \tau_{xy}^*}{\partial x}+\frac{\partial \sigma_y^*}{\partial y}+\frac{\partial \tau_{yz}^*}{\partial z}\right)$$
$$+\frac{\partial u}{\partial z}\left(\frac{\partial \tau_{xz}^*}{\partial x}+\frac{\partial \tau_{yz}^*}{\partial y}+\frac{\partial \sigma_z^*}{\partial z}\right)+\frac{\partial^2 u}{\partial x^2}\sigma_x^*+\frac{\partial^2 u}{\partial y^2}\sigma_y^*+\frac{\partial^2 u}{\partial z^2}\sigma_z^*$$
$$+2\frac{\partial^2 u}{\partial x\partial y}\tau_{xy}^*+2\frac{\partial^2 u}{\partial x\partial z}\tau_{xz}^*+2\frac{\partial^2 u}{\partial z\partial y}\tau_{yz}^*+\frac{V^*}{V}F_a^*=0$$

$$(76)\quad \frac{\partial v}{\partial x}\left(\frac{\partial \sigma_x^*}{\partial x}+\frac{\partial \tau_{xy}^*}{\partial y}+\frac{\partial \tau_{xz}^*}{\partial z}\right)+\left(1+\frac{\partial v}{\partial y}\right)\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y^*}{\partial y}+\frac{\partial \tau_{yz}^*}{\partial z}\right)$$
$$+\frac{\partial v}{\partial z}\left(\frac{\partial \tau_{xz}^*}{\partial x}+\frac{\partial \tau_{yz}^*}{\partial y}+\frac{\partial \sigma_z^*}{\partial z}\right)+\frac{\partial^2 v}{\partial x^2}\sigma_x^*+\frac{\partial^2 v}{\partial y^2}\sigma_y^*+\frac{\partial^2 v}{\partial z^2}\sigma_z^*$$
$$+2\frac{\partial^2 v}{\partial x\partial y}\tau_{xy}^*+2\frac{\partial^2 v}{\partial x\partial z}\tau_{xz}^*+2\frac{\partial^2 v}{\partial y\partial z}\tau_{yz}+\frac{V^*}{V}F_b^*=0$$

$$(77)\quad \frac{\partial w}{\partial x}\left(\frac{\partial \sigma_x^*}{\partial x}+\frac{\partial \tau_{xy}^*}{\partial y}+\frac{\partial \tau_{xz}^*}{\partial z}\right)+\frac{\partial w}{\partial y}\left(\frac{\partial \tau_{xy}^*}{\partial x}+\frac{\partial \sigma_y^*}{\partial y}+\frac{\partial \tau_{yz}^*}{\partial z}\right)$$
$$+\left(1+\frac{\partial w}{\partial z}\right)\left(\frac{\partial \tau_{xz}^*}{\partial x}+\frac{\partial \tau_{yz}^*}{\partial y}+\frac{\partial \sigma_z^*}{\partial z}\right)+\frac{\partial^2 w}{\partial x^2}\sigma_x^*+\frac{\partial^2 w}{\partial y^2}\sigma_y^*+\frac{\partial^2 w}{\partial z^2}\sigma_z^*$$
$$+2\frac{\partial^2 w}{\partial x\partial y}\tau_{xy}^*+2\frac{\partial^2 w}{\partial x\partial z}\tau_{xz}^*+2\frac{\partial^2 w}{\partial y\partial z}\tau_{yz}^*+\frac{V^*}{V}F_c^*=0$$

The above relations clearly show the interdependence between the rotation and the stresses in the equations of equilibrium in the general case. However, the above equations are in general far too complicated to be of significant practical value.

Since the basic assumption of this thesis is that the large deflection plate problem is both physically and geometrically non linear, the classical equations of equilibrium can't be used without a further analysis since their derivation is based upon

(1) The assumption of plane stress

(2) Geometric non linearity without regard to the physical non linearity.

The classical equilibrium equations (appendix C) are valid approximations of the more general equations (75), (76) and (77) and thus, the geometrical portion of their derivation remains valid for the purposes of this thesis. Thus, the equation

$$(78)\quad \frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\left(q + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} - X\frac{\partial w}{\partial x} - Y\frac{\partial w}{\partial y}\right)$$

remains valid, and the moments per unit length and twisting moments still are given by

$$(79)\quad M_x = \int_{-t/2}^{t/2} \sigma_x z \, dz$$

$$(80)\quad M_y = \int_{-t/2}^{t/2} \sigma_y z \, dz$$

$$(81)\quad M_{xy} = -\int_{-t/2}^{t/2} \tau_{xy} z \, dz$$

At this point, however, a departure from the classical approach becomes necessary. Instead of using the linearized stress relations to obtain a relationship between moments and deflection w, we will use the non linear form developed in appendix B, i.e.

(7) $$T = \lambda I_1 E_3 + 2G\eta + \left[(l-\lambda) I_1^2 - 2m I_2\right] E_3 + 2(m+\lambda-G) I_1 \eta + n\,\mathrm{Co}\,\eta + 4G\eta^2$$

Using the analogy between the stress-strain tensor equation and the moment-curvature tensor equation as suggested by Borg [5] we have:

(82) $$M = \lambda \mathcal{J}_{1c} E_3 + 2G\mathcal{R} + \left[(l-\lambda)\mathcal{J}_{1c}^2 - 2m\mathcal{J}_{2c}\right] E_3 + 2(m+\lambda-G)\mathcal{J}_{1c}\mathcal{R} + n\,\mathrm{Co}\,\mathcal{R} + 4G\mathcal{R}^2$$

where M is the moment per unit length tensor

$$M = \begin{pmatrix} M_{aa} & -M_{ab} & -Q_a C \\ M_{ba} & M_{bb} & -Q_b C \\ -Q_a C & -Q_b C & -\dfrac{gC^2}{2} \end{pmatrix}$$

and $\mathcal{R}$ is the curvature tensor given by

(83) $$\mathcal{R} = \begin{pmatrix} -\dfrac{1}{r_{aa}} & \dfrac{1}{r_{ab}} & \dfrac{1}{r_{ac}} \\ \dfrac{1}{r_{ba}} & -\dfrac{1}{r_{bb}} & \dfrac{1}{r_{bc}} \\ \dfrac{1}{r_{ca}} & \dfrac{1}{r_{cb}} & -\dfrac{1}{r_{cc}} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial}{\partial a}\left(\dfrac{\partial w}{\partial a}\right) & \dfrac{1}{2}\left[\dfrac{\partial}{\partial a}\left(\dfrac{\partial w}{\partial b}\right)+\dfrac{\partial}{\partial b}\left(\dfrac{\partial w}{\partial a}\right)\right] & \dfrac{1}{2}\left[\dfrac{\partial}{\partial a}\left(\dfrac{\partial w}{\partial c}\right)+\dfrac{\partial}{\partial c}\left(\dfrac{\partial w}{\partial a}\right)\right] \\ \dfrac{1}{2}\left[\dfrac{\partial}{\partial b}\left(\dfrac{\partial w}{\partial a}\right)+\dfrac{\partial}{\partial a}\left(\dfrac{\partial w}{\partial b}\right)\right] & \dfrac{\partial}{\partial b}\left(\dfrac{\partial w}{\partial b}\right) & \dfrac{1}{2}\left[\dfrac{\partial}{\partial b}\left(\dfrac{\partial w}{\partial c}\right)+\dfrac{\partial}{\partial c}\left(\dfrac{\partial w}{\partial b}\right)\right] \\ \dfrac{1}{2}\left[\dfrac{\partial}{\partial c}\left(\dfrac{\partial w}{\partial a}\right)+\dfrac{\partial}{\partial a}\left(\dfrac{\partial w}{\partial c}\right)\right] & \dfrac{1}{2}\left[\dfrac{\partial}{\partial c}\left(\dfrac{\partial w}{\partial b}\right)+\dfrac{\partial}{\partial b}\left(\dfrac{\partial w}{\partial c}\right)\right] & \dfrac{\partial}{\partial c}\left(\dfrac{\partial w}{\partial c}\right) \end{pmatrix}$$

$$+\frac{t}{4}\begin{pmatrix} \left(\dfrac{\partial^2 w}{\partial a^2}\right)^2+\left(\dfrac{\partial^2 w}{\partial a\partial b}\right)^2+\left(\dfrac{\partial^2 w}{\partial a\partial c}\right)^2 & \dfrac{\partial^2 w}{\partial a^2}\dfrac{\partial^2 w}{\partial a\partial b}+\dfrac{\partial^2 w}{\partial a\partial b}\dfrac{\partial^2 w}{\partial b^2}+\dfrac{\partial^2 w}{\partial a\partial c}\dfrac{\partial^2 w}{\partial b\partial c} & \dfrac{\partial^2 w}{\partial a^2}\dfrac{\partial^2 w}{\partial a\partial c}+\dfrac{\partial^2 w}{\partial a\partial b}\dfrac{\partial^2 w}{\partial b\partial c}+\dfrac{\partial^2 w}{\partial a\partial c}\dfrac{\partial^2 w}{\partial c^2} \\ \dfrac{\partial^2 w}{\partial a^2}\dfrac{\partial^2 w}{\partial a\partial b}+\dfrac{\partial^2 w}{\partial a\partial b}\dfrac{\partial^2 w}{\partial b^2}+\dfrac{\partial^2 w}{\partial a\partial c}\dfrac{\partial^2 w}{\partial c\partial b} & \left(\dfrac{\partial^2 w}{\partial b^2}\right)^2+\left(\dfrac{\partial^2 w}{\partial a\partial b}\right)^2+\left(\dfrac{\partial^2 w}{\partial b\partial c}\right)^2 & \dfrac{\partial^2 w}{\partial b\partial c}\dfrac{\partial^2 w}{\partial c\partial a}+\dfrac{\partial^2 w}{\partial b^2}\dfrac{\partial^2 w}{\partial c\partial b}+\dfrac{\partial^2 w}{\partial b\partial c}\dfrac{\partial^2 w}{\partial c^2} \\ \dfrac{\partial^2 w}{\partial a^2}\dfrac{\partial^2 w}{\partial a\partial c}+\dfrac{\partial^2 w}{\partial a\partial b}\dfrac{\partial^2 w}{\partial b\partial c}+\dfrac{\partial^2 w}{\partial a\partial c}\dfrac{\partial^2 w}{\partial c^2} & \dfrac{\partial^2 w}{\partial b\partial c}\dfrac{\partial^2 w}{\partial c\partial a}+\dfrac{\partial^2 w}{\partial b^2}\dfrac{\partial^2 w}{\partial c\partial b}+\dfrac{\partial^2 w}{\partial b\partial c}\dfrac{\partial^2 w}{\partial c^2} & \left(\dfrac{\partial^2 w}{\partial a\partial c}\right)^2+\left(\dfrac{\partial^2 w}{\partial b\partial c}\right)^2+\left(\dfrac{\partial^2 w}{\partial c^2}\right)^2 \end{pmatrix}$$

If we now use the classical large deflection theory assumption that the slope is small and the square is negligible compared to unity, and assuming Eulerian Coordinates, we have:

$$(84)\quad \mathcal{R} = \begin{pmatrix} -\frac{1}{r_x} & \frac{1}{r_{xy}} & \frac{1}{r_{xz}} \\ \frac{1}{r_{yx}} & -\frac{1}{r_y} & \frac{1}{r_{yz}} \\ \frac{1}{r_{zx}} & \frac{1}{r_{zy}} & -\frac{1}{r_z} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x \partial y} & \frac{\partial^2 w}{\partial x \partial z} \\ \frac{\partial^2 w}{\partial y \partial x} & \frac{\partial^2 w}{\partial y^2} & \frac{\partial^2 w}{\partial y \partial z} \\ \frac{\partial^2 w}{\partial z \partial x} & \frac{\partial^2 w}{\partial z \partial y} & \frac{\partial^2 w}{\partial z^2} \end{pmatrix}$$

At this point, it is necessary to consider what actually happens to the plate when we make the assumption $\frac{\partial w}{\partial z} \neq 0$ . This assumption, in effect, assumes a deformation gradient through the thickness of the plate (Poisson's effect) which alters the limits of integration in the expression for the moments per unit length. Thus, to be exact, for the assumption $\frac{\partial w}{\partial z} \neq 0$ we should write:

$$(85)\qquad M_x = \int_{-t/2-\delta}^{t/2+\delta} \sigma_x \, z \, dz$$

$$(86)\qquad M_y = \int_{-t/2-\delta}^{t/2+\delta} \sigma_y \, z \, dz$$

$$(87)\qquad M_{xy} = \int_{-t/2-\delta}^{t/2+\delta} \tau_{xy} \, z \, dz$$

where $\delta$ is the change of the neutral surface due to the deformation gradient $\frac{\partial w}{\partial z}$ across the thickness of the plate as shown in figure 2 below.

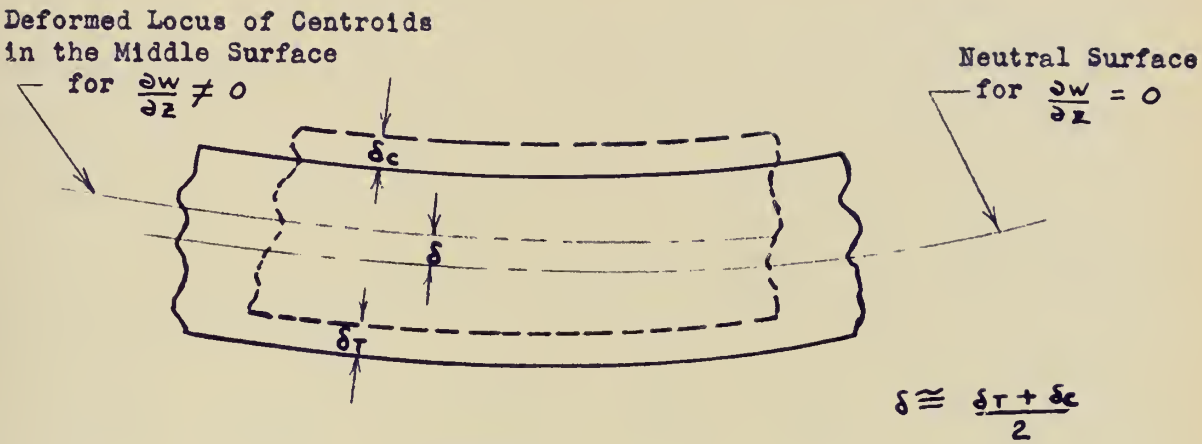



FIGURE 2

From figure 2, it can be seen that the total thickness is essentially not altered by either the assumption $\frac{\partial w}{\partial z} \neq 0$ or $\frac{\partial w}{\partial z} = 0$, but that the plate deflection $w$ is altered in the case of $\frac{\partial w}{\partial z} = 0$ by an amount $\delta$, which in effect, requires an additional constraining force $F(z)$ such that the total deflection is now $W = W(x,y) + w(\delta)$. Thus $\delta$ is in effect, a measure of the error of the assumption $\frac{\partial w}{\partial z} = 0$

A complete investigation of the effect of the $\delta$ factor is beyond the scope of this thesis, however, it can be seen from equation (84) for the curvature tensor, that the curvature elements containing $\frac{\partial w}{\partial z}$ are all of the order of magnitude $\epsilon^4$ as discussed on page 13. Accordingly, to the degree of accuracy of the other equations of this thesis, we can write the curvature tensor as

$$(88) \qquad \mathcal{R} = \begin{pmatrix} \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x \partial y} & 0 \\ \frac{\partial^2 w}{\partial y \partial x} & \frac{\partial^2 w}{\partial y^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{r_x} & \frac{1}{r_{xy}} & \frac{1}{r_{xz}} \\ \frac{1}{r_{yx}} & -\frac{1}{r_y} & \frac{1}{r_{yz}} \\ \frac{1}{r_{zx}} & \frac{1}{r_{zy}} & -\frac{1}{r_z} \end{pmatrix}$$

From equation (88), we can see that we are left with the expressions for curvature that are identical to those contained in the classical theory in two dimensional form, where the third dimension terms are assumed to be zero.

From the curvature tensor [equation (88)] we have the following relations

(89) $\quad \mathcal{J}_{1c} = -\left(\frac{1}{r_x} + \frac{1}{r_y} + \frac{1}{r_z}\right) = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$

(90) $\quad \mathcal{J}_{2c} = \frac{1}{r_x}\frac{1}{r_y} - \left(\frac{1}{r_{xy}}\right)^2$ + terms involving $\frac{1}{r_z}, \frac{1}{r_{xz}}$ and $\frac{1}{r_{yz}}$

$$= \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$$

(91) $\quad \mathcal{J}_{1c}^{\,2}$ contains terms of higher order than desired and thus can be neglected

(92)
$$C_o \mathcal{R} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \end{pmatrix}$$

= Terms of higher order than desired ($\epsilon^4$)

$\mathcal{R}\mathcal{R}$ = terms of the order $\epsilon^4$ and can be neglected

$\mathcal{J}_{lc}\mathcal{R}$ terms of the order $\epsilon^4$ and can be neglected

Thus, we have, retaining terms to the order $\epsilon^2$

$$(93) \quad M = \lambda\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + 2G\begin{pmatrix}\frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x\partial y} & 0\\ \frac{\partial^2 w}{\partial x\partial y} & \frac{\partial^2 w}{\partial y^2} & 0\\ 0 & 0 & 0\end{pmatrix}$$

It should be noted at this point that retaining only those terms to the order $\epsilon^2$ reduces the Moment-Curvature tensor to a form that is analogous to the linearized Hooke's law $T = \lambda I_1 E_3 + 2G\eta$ . Thus, we can see that the classical assumptions in effect force the equilibrium equations into a case of geometric non linearity and physical linearity where the third order elastic constants do not appear. This fact is born out by the fact that the third order constants l, m and n are multipliers of non linear terms and cannot appear under the classical assumptions for the equilibrium equation.

Inasmuch as we are now left with a linearized stress-strain relation and its analogous Moment-Curvature relation, the classical equilibrium equation derived on the basis of plane-stress becomes valid for cases where the actual case approximates plane-stress. Thus we have, from appendix C and reference [18] (neglecting body forces)

$$(94) \quad \nabla^4 w = \frac{p}{D} + \frac{t}{D}\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}\right)$$

(PLANE STRESS)

Proceeding with the development for the assumption $\frac{\partial w}{\partial z} = 0$ we have, from $e = \frac{z}{r}$ in the two dimensional case and the analogy between tensor equations

$$(95) \qquad \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{pmatrix} = -z\lambda\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)E_2 - 2Gz\begin{pmatrix} \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x \partial y} \\ \frac{\partial^2 w}{\partial y \partial x} & \frac{\partial^2 w}{\partial y^2} \end{pmatrix}$$

Thus

$$(96) \qquad \sigma_x = -z\left[\lambda\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + 2G\frac{\partial^2 w}{\partial x^2}\right]$$

$$(97) \qquad \sigma_y = -z\left[\lambda\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + 2G\frac{\partial^2 w}{\partial y^2}\right]$$

$$(98) \qquad \tau_{xy} = -z\ 2G\frac{\partial^2 w}{\partial x \partial y}$$

Substituting equations (96), (97) and (98) into the expressions for moments per unit length, we have

$$(99) \qquad M_x = \int_{-t/2}^{t/2} -z^2\left[\lambda\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + 2G\frac{\partial^2 w}{\partial x^2}\right]dz$$

$$(100) \qquad M_y = \int_{-t/2}^{t/2} -z^2\left[\lambda\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + 2G\frac{\partial^2 w}{\partial y^2}\right]dz$$

$$(101) \qquad M_{xy} = \int_{-t/2}^{t/2} +z^2\ 2G\frac{\partial^2 w}{\partial x \partial y}\,dz$$

Performing the above integration, we now have:

$$(102) \qquad M_x = -\frac{t^3}{12}\left[\lambda\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + 2G\frac{\partial^2 w}{\partial x^2}\right]$$

$$(103) \qquad M_y = -\frac{t^3}{12}\left[\lambda\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + 2G\frac{\partial^2 w}{\partial y^2}\right]$$

(104) $$M_{xy} = + \frac{t^3}{12}\left(2G\frac{\partial^2 w}{\partial x \partial y}\right)$$

Differentiating equations (102), (103) and (104) to obtain

$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2}$ , we have

(105) $$\frac{\partial^2 M_x}{\partial x^2} = -\frac{t^3}{12}\left[\lambda\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2}\right) + 2G\frac{\partial^4 w}{\partial x^4}\right]$$

(106) $$\frac{\partial^2 M_y}{\partial y^2} = -\frac{t^3}{12}\left[\lambda\left(\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + 2G\frac{\partial^4 w}{\partial y^4}\right]$$

(107) $$-2\frac{\partial^2 M_{xy}}{\partial x \partial y} = -\frac{t^3}{12}\left[4G\frac{\partial^4 w}{\partial x^2 \partial y^2}\right]$$

adding, we have

(108) $$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\frac{t^3}{12}(\lambda + 2G)\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right)$$

$$= -\frac{t^3}{12}(\lambda + 2G)\nabla^4 w$$

Using the relationship $\lambda + 2G = \frac{2G\nu}{1-2\nu} + 2G = \frac{2G(1-\nu)}{1-2\nu}$

and $G = \frac{E}{2(1+\nu)}$ , we have $\lambda + 2G = \frac{E(1-\nu)^2}{(1-\nu^2)(1-2\nu)}$

Since the flexural rigidity is $D = \frac{Et^3}{12(1-\nu^2)}$

We have finally

(109) $$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -D\frac{(1-\nu)^2}{(1-2\nu)}\nabla^4 w$$

The equation of equilibrium for the assumption $\frac{\partial w}{\partial z} = 0$ now becomes (neglecting body forces)

$$(110) \quad D\frac{(1-\nu)^2}{(1-2\nu)} \nabla^4 W = p + N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}$$

or from the stress function F defined as

$$(111) \quad \frac{N_x}{t} = \sigma_x = \frac{\partial^2 F}{\partial y^2}$$

$$(112) \quad \frac{N_y}{t} = \sigma_y = \frac{\partial^2 F}{\partial x^2}$$

$$(113) \quad \frac{N_{xy}}{t} = \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

We have

$$(114) \quad \nabla^4 W = \frac{(1-2\nu)}{(1-\nu)^2} \frac{p}{D} + \frac{t}{D} \frac{(1-2\nu)}{(1-\nu)^2} \left( \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right)$$

As a check on the above equation, since the basis was a linearized tensor equation, we should be able to obtain the identical expression using the linearized plane strain equations resulting from the linearized Hooke's law

$T = \lambda I_1 E_3 + 2G\eta$ where for plane strain we have the elements $e_z = \gamma_{yz} = \gamma_{xz} = \frac{\partial}{\partial z} = 0$

Thus, we have the expressions

$$(115) \quad \sigma_x = \lambda(e_x + e_y) + 2G e_x$$

$$(116) \quad \sigma_y = \lambda(e_y + e_x) + 2G e_y$$

$$(117) \quad \tau_{xy} = \tau_{yx} = G \gamma_{xy}$$

Since, in the linear relations

$$\gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -2z\frac{\partial^2 w}{\partial x \partial y} \qquad \text{where} \quad u = -z\frac{\partial w}{\partial x} \;\; \& \;\; v = -z\frac{\partial w}{\partial y}$$

and $e_x = \frac{z}{r_x} = -z\frac{\partial^2 w}{\partial x^2}$

$$e_y = \frac{z}{r_y} = -z\frac{\partial^2 w}{\partial y^2}$$

we have the expressions

(118) $\sigma_x = -\lambda z\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - 2Gz\frac{\partial^2 w}{\partial x^2}$

(119) $\sigma_y = -\lambda z\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) - 2Gz\frac{\partial^2 w}{\partial y^2}$

(120) $\tau_{xy} = -2Gz\left(\frac{\partial^2 w}{\partial x \partial y}\right)$

Equations (118), (119) and (120) are identical to equations (96), (97) and (98), thus we could have derived the equilibrium equation (114) directly by the plane strain linear relations. The point to be stressed in the derivation of equation (114) is that the classical large deflection assumptions, when coupled with the assumption $\frac{\partial w}{\partial z} = 0$ leads to a plane strain equilibrium equation rather than a plane stress equation as derived by von Kármán.

In summary, within the limitations of the assumptions of classical large deflection plate problems, we now have two equilibrium

equations, the von Kármán equation where the derivation is based on the linearized Hooke's law with strain a function of stress and the equation derived in this thesis with stress a function of strain. Thus we have an upper and lower bound on the assumption of plane stress or plane strain as follows:

PLANE STRESS (Neglecting Body Forces)

$$\nabla^4 W = \frac{p}{D} + \frac{t}{D}\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 W}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 W}{\partial x \partial y}\right) \tag{94}$$

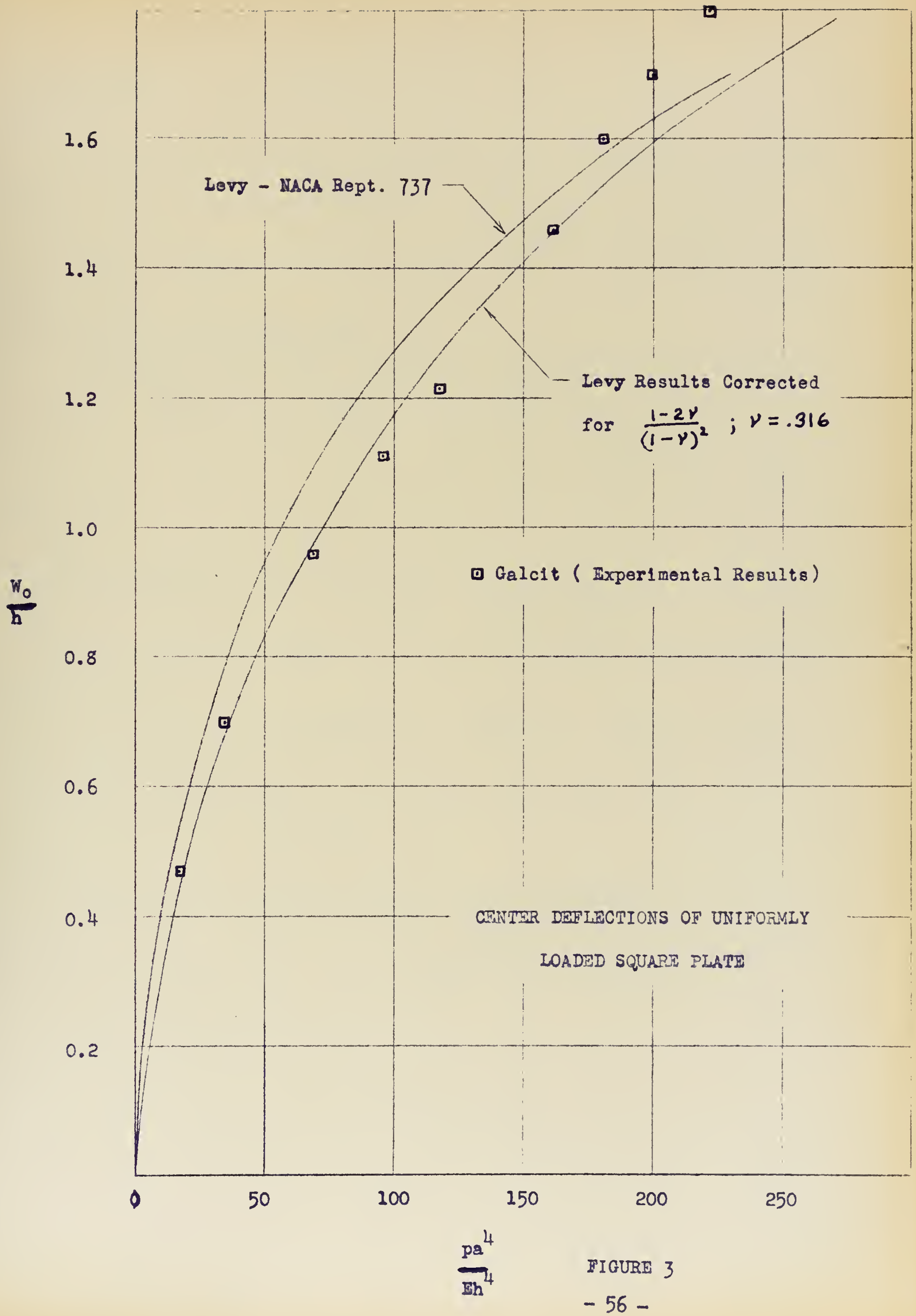
PLANE STRAIN (Neglecting Body Forces)

$$\nabla^4 W = \frac{(1-2\nu)}{(1-\nu)^2}\frac{p}{D} + \frac{t}{D}\frac{(1-2\nu)}{(1-\nu)^2}\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 W}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 W}{\partial x \partial y}\right) \tag{114}$$

As previously discussed, the actual case will be somewhere between the limits of the above equations.

Applying the correction $\frac{1-2\nu}{(1-\nu)^2}$ as a linear correction to the flexural rigidity in the data reported in references [14] and [19] and comparing the results with experimental results shown in reference [19] we can see by figure 3 that the correction appears to be about the correct order of magnitude and in the correct direction. Thus, for this case, it would appear that the actual case is closer to plane strain than to plane stress. On the other hand, one must be cautious about drawing hasty conclusions based upon figure 3 since there was insufficient data available in reference [19] to accurately evaluate the boundary conditions of the experimental results. Further discussion of the implications

of the correction of Levy's solution and the divergence of both theoretical solutions from the Galcit experimental results is contained on page 62.



1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
$\frac{W_o}{h}$
0
50
100
150
200
250
$\frac{pa^4}{Eh^4}$
Levy - NACA Rept. 737
Levy Results Corrected for $\frac{1-2\nu}{(1-\nu)^2}$ ; $\nu = .316$
Galcit ( Experimental Results)
CENTER DEFLECTIONS OF UNIFORMLY LOADED SQUARE PLATE


FIGURE 3

## CONCLUSIONS

In addition to the classical large deflection thin plate equations derived by von Kármán, we now have three new compatibility equations and one new equilibrium equation based on the assumptions indicated below:

A. COMPATIBILITY EQUATIONS

Classical von Kármán Compatibility Equation [11]

(a) $\nabla^4 F = -E\,\mathcal{J}_{2C} = -(\nu+1)\,2G\,\mathcal{J}_{2C}$

Based on $\sigma_z = 0$, linear stress-strain relation and plane stress

Borg Compatibility Equations [6]

(b) $\nabla^4 F = -\left(G + \frac{n}{4}\right)\mathcal{J}_{1LD} - \left(4G + \frac{n}{2}\right)\mathcal{J}_{2C}$

Based on $\sigma_z = 0$, $\frac{\partial w}{\partial z} = 0$ and a non linear stress-strain relation

(c) $\nabla^4 F = \left[\frac{m}{2}\frac{(1-2\nu)}{(1-\nu)} - \frac{n}{4} - \frac{G\nu}{1-\nu}\right]\mathcal{J}_{1LD} - \frac{1}{1-\nu}\left(4G + \frac{n}{2}\right)\mathcal{J}_{2C}$

Based on $\frac{\partial w}{\partial z} = 0$, no restriction on $\sigma_z$ and a non linear stress-strain relation.

Compatibility Equation Developed in this Thesis

(d) $\nabla^4 F = \left[\frac{m}{2}(1-2\nu) - \frac{n}{4} - 2G\nu\right]\mathcal{J}_{1LD} - (\nu+1)\left(4G + \frac{n}{2}\right)\mathcal{J}_{2C}$

Based on $\sigma_z = 0$, no restriction on $\frac{\partial w}{\partial z}$ and a non linear stress-strain relation.

B. EQUILIBRIUM EQUATIONS

Classical von Kárman Equilibrium Equation [11]

(e) $$\nabla^4 w = \frac{p}{D} + \frac{t}{D}\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right)$$

Based on a linearized plane stress assumption (Assumes strain is a function of stress).

Equilibrium Equation Developed in this Thesis

(f) $$\nabla^4 w = \frac{1-2\nu}{(1-\nu)^2}\frac{p}{D} + \frac{t}{D}\frac{(1-2\nu)}{(1-\nu)^2}\left(\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right)$$

Based on non-linear development approaching plane strain, but linearized by use of the classical large deflection assumptions. (Assumes atress is a function of strain).

From the above four compatibility equations and two equilibrium equations, we can now tailor the equation to suit a particular large deflection plate problem dependent upon the degree of accuracy required and the degree and type of deflection or loading involved. In other words, the equations can be tailored as follows:

(1) Equation (a), or the classical compatibility equation can be used where the problem is linear physically, but non linear geometrically (moderately large deflections), or where the degree of accuracy is not critical.

(2) Equation (b) can be used for very thin plates where the problem is non linear both physically and geometrically, but where the Poisson's effect across the thickness can be ignored, and the assumption $\sigma_z = 0$ is valid.

(3) Equation (c) can be used for thin plates where the

problem is non-linear both physically and geometrically, but where the Poisson's effect across the thickness can be ignored, and where the normal ($\sigma_z$) stresses can no longer be neglected.

(4) Equation (d) can be used for moderately thick plate problems non linear both physically and geometrically, and where the Poisson's effect across the thickness of the plate must be included, but where the assumption $\sigma_z = 0$ still remains valid (Thicker than Thin Plate Theory).

(5) Equation (e), or the classical large deflection equilibrium equation, can be used in cases where the boundary conditions and loading are such that the actual case is closer to plane stress than plane strain.

(6) Equilibrium equation (f) can be used for problems where a plane stress assumption is not valid and where the Poisson's effect across the thickness of the plate can be ignored ($\frac{\partial w}{\partial z} = 0$).

Reflecting on the above equations, the question arises: why do the von Kármán equations predict results that often closely approximate experimental results, but occasionally are significantly at variance? Particularly interesting is the fact that for the preliminary solutions Borg has achieved with equation (b) for circular plates with circular symmetry, the results are very close to those obtained with the classical von Kármán equations (a) and (e) [7]. In view of the apparent wide variation in equations (b) and (a), one must ask - why?

If we neglect the term in equation (d) involving the large deflection term $\mathcal{G}_{ILD}$ , we can note the similarity to the classical von Kármán equation i.e.

(a) $\nabla^4 F = -(\nu+1)\ 2G\ \mathcal{G}_{2c}$

(d) $\nabla^4 F_{\text{modified}} = -(\nu+1)(4G+\frac{n}{2})\mathcal{G}_{2c}$

From equation (a) and $(d)_{\text{modified}}$, we can see that if the third order elastic constant n is equal to -4G, the von Karman equation is identical to equation (d) if the term $\left[\frac{m}{2}(1-2\nu)-\frac{n}{4}-2G\nu\right]\mathcal{G}_{ILD}$ can be neglected.

As shown in Table I, Borg obtained values for n (corresponding to Young's Modulus of 30 x $10^6$ ) of -34.2 x $10^6$ . Also, Smith obtained values for Austenitic Steel of -40.0 x $10^6$ and for 0.6 Carbon Steel of -67 x $10^6$ . If we assume an average of the above values of n = -47 x $10^6$ and the commonly accepted value of G = 11.5 x $10^6$ , or 4G = 46 x $10^6$ , we can see that the actual numerical value of n probably does in fact lie very close to - 4G. Hence, we can approximate equation (d) as:

$$\nabla^4 F \cong \left[\left(\frac{m}{2}+G\right)(1-2\nu)\right]\mathcal{G}_{ILD} - (\nu+1)\ 2G\,\mathcal{G}_{2c}$$

The above equation still does not explain why the von Kármán equation (a) provides valid results in the majority of the cases since we have only explained the right term and not the term involving $\mathcal{G}_{ILD}$ .

If we analyze the function $\mathcal{G}_{ILD} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left[\left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]$ by vector algebra, we can see that grad $\vec{w} = \frac{\partial w}{\partial x} i + \frac{\partial w}{\partial y} j = \nabla \vec{w}$ and hence grad $\vec{w}$ · grad $\vec{w}$ $= \left(\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2 = |\nabla \vec{w}|^2$ but the maximum directional derivative $\frac{dw}{ds}_{max} = |\text{grad}\ \vec{w}|$

$$\left|\frac{dw}{ds_{max}}\right|^2 = \left|\,\mathrm{grad}\ \vec{w}\right|^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

hence, the vector meaning of $\mathcal{J}_{ILD}$ is that it refers to the Laplacian of the square of the maximum directional derivative, i.e. $\nabla^2 \left|\frac{dw}{ds_{max}}\right|^2$

But in the classical large deflection assumptions, the square of the slope was considered small compared to unity, therefore the Laplacian of the square of the directional derivative will be much smaller than unity. Thus, within the limits of the classical assumptions for large deflection plate problems, the function involving $\mathcal{J}_{ILD}$ will be negligible unless the slope, and hence, the deflection is very large. Stated another way, when the deflections become such that the term involving $\mathcal{J}_{ILD}$ cannot be ignored, the classical assumption, in which the square of the slope is much smaller than one, is questionable.

If we further consider the elastic constants associated with $\mathcal{J}_{ILD}$ i.e. $\left[\frac{m}{2}(1-2\nu) - \frac{n}{4} - 2G\nu\right]$ and assume that n = -4G , we have $\nabla^4 F = \left(\frac{m}{2}+G\right)(1-2\nu)\,\mathcal{J}_{ILD} - (\nu+1)\,2G\,\mathcal{J}_{2c}$

From Table I, for Borg's and Smith's values of m, we can assume an average value for steel of approximately m = -60 x $10^6$ , or about -5.2G. With this assumption, we can approximate the compatibility equation (d) as $-(1.6G)(1-2\nu)\,\mathcal{J}_{ILD} - (\nu+1)\,2G\,\mathcal{J}_{2c}$

or, for $\nu = 0.3$ , $\nabla^4 F = -.64 G \mathcal{g}_{1LD} - 2.6 G \mathcal{g}_{2c}$ or

$$\nabla^4 F = -G \left[ .64 \mathcal{g}_{1LD} + 2.6 \mathcal{g}_{2c} \right]$$

Hence, we can see that the elastic constant contribution to the terms of the classical equation is about 4 times as great as the contribution to the term involving $\mathcal{g}_{1LD}$ .

From Figure 3, it can be seen that experimental values for the simply supported square plate tend to diverge from theoretical values above $\frac{W_o}{t}$ of about 1.5 for the data presented. Thus, from the hypothesis regarding the contribution of the term involving $\mathcal{g}_{1LD}$ and the results shown in figure 3, we can further hypothesize that for very large deflections ( $\frac{W_o}{t}$ greater than about 1.5) the term involving $\mathcal{g}_{1LD}$ begins to become significant and the von Kármán equations lead to increasing error with increasing deflection. Below the divergence area, the $\mathcal{g}_{1LD}$ term is not significant and the von Kármán equation is very close to the more exact equation (d).

The classical equation of equilibrium (e) is identical in form to the equilibrium equation derived in this thesis (f), with the exception of the coefficient involving Poisson's ratio $\frac{(1-2\nu)}{(1-\nu)^2}$ . For a Poisson's ratio of .3 the coefficient is 0.816 and for $\nu$ = .25 , the coefficient is 0.89. However, as the deflections become very large and the deformations approach the plastic range, we have a different situation. In classical theory, when an isotropic material reaches plastic deformation,

Poisson's ratio is taken to be $\frac{1}{2}$ [23]. However the coefficient $\frac{1-2\nu}{(1-\nu)^2}$ goes to zero at $\nu = \frac{1}{2}$ and hence, equation (f) becomes $\nabla^4 w = 0$. Reference [25] states that some investigators have obtained values of Poisson's ratio in the plastic range much greater than $\frac{1}{2}$ (as high as 0.8) which leads to further questions since the equilibrium equation (f) becomes $\nabla^4 w = -f\,(F,w,x,y)$. As a result, we are led to the tentative conclusion that equation (f) is only valid in the elastic range where the range of $\nu$ is 0.25 to 0.35, or that for deflections approaching the plastic range, we can no longer ignore the Poisson's effect across the thickness of the plate (the assumption $\frac{\partial w}{\partial z} = 0$ is not valid) and a three dimensional development similar to the development of equation (f) with $\frac{\partial w}{\partial z} \neq 0$ will be required. It should be noted that the three dimensional development of an equilibrium equation where $\frac{\partial w}{\partial z} \neq 0$, as proposed, would be based on the assumption that stress is a function of strain as opposed to the classical assumption of strain is a function of stress.

One final point to be noted in the discussion of this thesis is that in equation (d)

$$\text{(d)} \quad \nabla^4 F = \left[\frac{m}{2}(1-2\nu) - \frac{n}{4} - 2G\nu\right] \mathcal{J}_{1LD} - (\nu+1)\left(4G + \frac{n}{2}\right) \mathcal{J}_{2C}$$

the derivation was based upon the classical large deflection plate problem assumptions, but including terms applicable to plates of finite thickness, and with finite deflection theory. As a result, the development involves a compromise between thick and thin plate theory. Hence, we have a third category of plate theory which may be called "Thicker than Thin Plate Theory".

## RECOMMENDATIONS

The work done so far in the application of finite deformation theory to structures has hardly scratched the surface of a wide and virgin field of science. To attempt a discussion of areas requiring further investigation would be the subject of a thesis in itself.

A few of the more obvious areas of interest to the structural engineer which require further investigation are:

(1) To obtain more reliable values of the third order elastic constants l, m, and n.

(2) To obtain solutions to large deflection plate problems by use of the equations developed in this thesis, and for which there are experimental results available by which one can obtain an idea of the significance of the third order constants, and to determine the validity of the hypothesis made in the conclusions of this thesis.

(3) Determine values of $\delta$ as applied to the limits of integration of the moment per unit length equations in order to determine the magnitude of the error involved in neglecting Poisson's effect across the thickness of the plate.

(4) Through dimensional analysis, attempt to determine a physical relationship for the third order elastic constants as has been done with Lame's constants in linear theory. In other words, attempt to obtain a relationship between l, m, and n, and the defining parameters of a material that determine its

thermodynamic solid state point (Temperature, Pressure, Specific Volume or Density, Conductivity, etc.). The long term implication of the above analysis is the development of a combination Solid State Physics/Thermodynamics approach in the determination of material properties. A logical extension would be to include time dependent functions for purposes of analysis of creep, fatigue, etc.

(5) Attempt to determine equations for critical stresses for buckling or instability using the plate strain tensor derived in this thesis with a linear stress-strain relation in addition to buckling equations using the compatibility equations of this thesis which include the effect of the third order elastic constants of the non linear stress-strain relation.

(6) Derive a Compatibility Equation under which no restriction is made on either $\sigma_z$ or $\frac{\partial w}{\partial z}$ ( $\sigma_z \neq 0$ and $\frac{\partial w}{\partial z} \neq 0$ ) for use in "Thicker than Thin Plate Theory" where $\sigma_z$ must not be neglected (as in the case of very deep-diving submarines).

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APPENDIX A

Strain Tensor $\eta$

In tensor notation, the state of strain of a body, initially unstressed where point "a", having coordinates (a,b,c), then strained to point "x" having coordinates (x,y,z), is given by: [5]

(A-1) STATE OF STRAIN = $dx^* dx - da^* da$

Since $dx = J da$, where $J = \dfrac{\partial(x,y,z)}{\partial(a,b,c)}$ Jacobian matrix of (x,a)

or $$J = \begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{pmatrix} = \text{a tensor}$$

and $dx^* = da^* J^*$

where $J^*$ is the transpose of the Jacobian

$$J^* = \begin{pmatrix} \frac{\partial}{\partial a} \\ \frac{\partial}{\partial b} \\ \frac{\partial}{\partial c} \end{pmatrix} (x \quad y \quad z) = \begin{pmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{pmatrix}$$

and is also a tensor, we can say:

(A-2) STATE OF STRAIN = $da^* (J^* J - E_3) da$

where $E_3$ is the 3 x 3 unit matrix

Since $x = a + u$, $y = b + v$, $z = c + w$

where $x = x(a,b,c)$, $y = y(a,b,c)$, and $z = z(a,b,c)$

we have $\frac{\partial x}{\partial a} = 1 + \frac{\partial u}{\partial a}$ , $\frac{\partial y}{\partial b} = 1 + \frac{\partial v}{\partial b}$ , $\frac{\partial z}{\partial c} = 1 + \frac{\partial w}{\partial c}$

$$\frac{\partial x}{\partial b} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial b} = \frac{\partial u}{\partial b}, \quad \frac{\partial y}{\partial a} = \frac{\partial v}{\partial y}\frac{\partial y}{\partial a} = \frac{\partial v}{\partial a}$$

$$\frac{\partial z}{\partial c} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial c} = \frac{\partial u}{\partial c}$$

etc. (using chain rule for other terms)

we can express Jacobian as:

(A-3)
$$J = \begin{pmatrix} 1+\frac{\partial u}{\partial a} & \frac{\partial u}{\partial b} & \frac{\partial u}{\partial c} \\ \frac{\partial v}{\partial a} & 1+\frac{\partial v}{\partial b} & \frac{\partial v}{\partial c} \\ \frac{\partial w}{\partial a} & \frac{\partial w}{\partial b} & 1+\frac{\partial w}{\partial c} \end{pmatrix}$$

and
(A-4)
$$J^* = \begin{pmatrix} 1+\frac{\partial u}{\partial a} & \frac{\partial v}{\partial a} & \frac{\partial w}{\partial a} \\ \frac{\partial u}{\partial b} & 1+\frac{\partial v}{\partial b} & \frac{\partial w}{\partial b} \\ \frac{\partial u}{\partial c} & \frac{\partial v}{\partial c} & 1+\frac{\partial w}{\partial c} \end{pmatrix}$$

Expanding $J^*J$ we have:

(A-5)
$$J^*J = \begin{pmatrix} \left[1+2\frac{\partial u}{\partial a}+\left(\frac{\partial u}{\partial a}\right)^2+\left(\frac{\partial v}{\partial a}\right)^2+\left(\frac{\partial w}{\partial a}\right)^2\right] & \left[\frac{\partial u}{\partial b}+\frac{\partial v}{\partial a}+\frac{\partial v}{\partial a}\frac{\partial v}{\partial b}+\frac{\partial u}{\partial a}\frac{\partial u}{\partial b}+\frac{\partial w}{\partial a}\frac{\partial w}{\partial b}\right] & \left[\frac{\partial u}{\partial c}+\frac{\partial w}{\partial a}+\frac{\partial v}{\partial a}\frac{\partial v}{\partial c}+\frac{\partial u}{\partial a}\frac{\partial u}{\partial c}+\frac{\partial w}{\partial a}\frac{\partial w}{\partial c}\right] \\ \left[\frac{\partial u}{\partial b}+\frac{\partial v}{\partial a}+\frac{\partial v}{\partial a}\frac{\partial v}{\partial b}+\frac{\partial u}{\partial a}\frac{\partial u}{\partial b}+\frac{\partial w}{\partial a}\frac{\partial w}{\partial b}\right] & \left[\left(\frac{\partial u}{\partial b}\right)^2+1+2\frac{\partial v}{\partial b}+\left(\frac{\partial v}{\partial b}\right)^2+\left(\frac{\partial w}{\partial b}\right)^2\right] & \left[\frac{\partial w}{\partial b}+\frac{\partial v}{\partial c}+\frac{\partial v}{\partial b}\frac{\partial v}{\partial c}+\frac{\partial u}{\partial b}\frac{\partial u}{\partial c}+\frac{\partial w}{\partial b}\frac{\partial w}{\partial c}\right] \\ \left[\frac{\partial u}{\partial c}+\frac{\partial w}{\partial a}+\frac{\partial v}{\partial a}\frac{\partial v}{\partial c}+\frac{\partial u}{\partial a}\frac{\partial u}{\partial c}+\frac{\partial w}{\partial a}\frac{\partial w}{\partial c}\right] & \left[\frac{\partial v}{\partial c}+\frac{\partial w}{\partial b}+\frac{\partial v}{\partial b}\frac{\partial v}{\partial c}+\frac{\partial u}{\partial b}\frac{\partial u}{\partial c}+\frac{\partial w}{\partial b}\frac{\partial w}{\partial c}\right] & \left[1+2\frac{\partial w}{\partial c}+\left(\frac{\partial u}{\partial c}\right)^2+\left(\frac{\partial v}{\partial c}\right)^2+\left(\frac{\partial w}{\partial c}\right)^2\right] \end{pmatrix}$$

a symmetric tensor

Writing above expression in terms of orders of tensors, we have:

$$J^*J = E_3 + 2\eta_1 + 2\eta_2 = E_3 + 2(\eta_1 + \eta_2) + E_3 + 2\eta$$

where $\eta_1$ is a 1st order tensor

and $\eta_2$ is a 2nd order tensor

$\eta$ is the combination 1st and 2nd order strain matrix

and rearranging, we have

$$\text{(A-7)} \quad \eta = \begin{pmatrix} \eta_{aa} & \eta_{ab} & \eta_{ac} \\ \eta_{ba} & \eta_{bb} & \eta_{bc} \\ \eta_{ca} & \eta_{cb} & \eta_{cc} \end{pmatrix} = \eta_1 + \eta_2 = \frac{J^*J - E_3}{2}$$

From the expansion of $J^*J$ and the relation $\eta = \frac{1}{2}\left(J^*J - E_3\right)$

we have

$$\text{(A-8)} \quad \eta = \begin{pmatrix} \frac{\partial u}{\partial a} & \frac{1}{2}\left(\frac{\partial u}{\partial b} + \frac{\partial v}{\partial a}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial c} + \frac{\partial w}{\partial a}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial b} + \frac{\partial v}{\partial a}\right) & \frac{\partial v}{\partial b} & \frac{1}{2}\left(\frac{\partial v}{\partial c} + \frac{\partial w}{\partial b}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial c} + \frac{\partial w}{\partial a}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial c} + \frac{\partial w}{\partial b}\right) & \frac{\partial w}{\partial c} \end{pmatrix}$$

$$+ \frac{1}{2} \begin{pmatrix} \left[\left(\frac{\partial u}{\partial a}\right)^2 + \left(\frac{\partial v}{\partial a}\right)^2 + \left(\frac{\partial w}{\partial a}\right)^2\right] & \left[\frac{\partial u}{\partial a}\frac{\partial u}{\partial b} + \frac{\partial v}{\partial a}\frac{\partial v}{\partial b} + \frac{\partial w}{\partial a}\frac{\partial w}{\partial b}\right] & \left[\frac{\partial u}{\partial a}\frac{\partial u}{\partial c} + \frac{\partial v}{\partial a}\frac{\partial v}{\partial c} + \frac{\partial w}{\partial a}\frac{\partial w}{\partial c}\right] \\ \left[\frac{\partial u}{\partial a}\frac{\partial u}{\partial b} + \frac{\partial v}{\partial a}\frac{\partial v}{\partial b} + \frac{\partial w}{\partial a}\frac{\partial w}{\partial b}\right] & \left[\left(\frac{\partial u}{\partial b}\right)^2 + \left(\frac{\partial v}{\partial b}\right)^2 + \left(\frac{\partial w}{\partial b}\right)^2\right] & \left[\frac{\partial u}{\partial b}\frac{\partial u}{\partial c} + \frac{\partial v}{\partial b}\frac{\partial v}{\partial c} + \frac{\partial w}{\partial b}\frac{\partial w}{\partial c}\right] \\ \left[\frac{\partial u}{\partial a}\frac{\partial u}{\partial c} + \frac{\partial v}{\partial a}\frac{\partial v}{\partial c} + \frac{\partial w}{\partial a}\frac{\partial w}{\partial c}\right] & \left[\frac{\partial u}{\partial b}\frac{\partial u}{\partial c} + \frac{\partial v}{\partial b}\frac{\partial v}{\partial c} + \frac{\partial w}{\partial b}\frac{\partial w}{\partial c}\right] & \left[\left(\frac{\partial u}{\partial c}\right)^2 + \left(\frac{\partial v}{\partial c}\right)^2 + \left(\frac{\partial w}{\partial c}\right)^2\right] \end{pmatrix}$$

The above expression expresses the strain tensor in the Lagrangian coordinate system where the movement of each particle in the body is followed (i.e. the initial coordinates of a particle were (a,b,c) then the final coordinates of the same particle in the deformed position at (x, y, z) is [ x(a,b,c), y(a,b,c), z(a,b,c)] ). [5] If the assumption is made that the deformation is sufficiently large that the 2nd order strain tensor must be included, but is sufficiently small that "a" is essentially the same as "x", the expression for strain can then be expressed in the Eulerian coordinate system (i.e. conditions are expressed at each point in the deformed body, and a particular point rather than a particular particle is considered and quantities considered are functions of x, y and z ).

Hence, we have the approximation

$$\eta = \begin{pmatrix} e_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & e_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & e_z \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{pmatrix} \tag{A-9}$$

$$+\frac{1}{2}\begin{pmatrix} \left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right] & \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] & \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right] \\ \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] & \left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right] & \left[\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right] \\ \left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right] & \left[\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right] & \left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right] \end{pmatrix}$$

(using American notation for elongation and shear strain).

APPENDIX B

<u>Derivation of Finite Deformation Stress-Strain Relation</u>

I. The fundamental relation of elasticity theory connecting stress and strain is:

(B-1) $$T = \left(\frac{\rho_x}{\rho_a}\right) J \frac{\partial \phi}{\partial \eta} J^* \qquad [1]$$

where T is the stress tensor

$$T = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}, \qquad J = \text{Jacobian} = \frac{\partial(x, y\, z)}{\partial(a, b, c)}$$

$\frac{\rho_x}{\rho_a}$ is the compression ratio $= \frac{1}{\det J}$

$\phi = \phi(\eta)$ = energy of deformation / unit initial vol. = $\rho_a \psi$

$\eta$ = strain matrix =

(B-2) $$\eta = \frac{1}{2}\left(J^* J - E_3\right)$$

(see Appendix A)

II. If the deformable medium is assumed to be isentropic, it is elastically insensitive to every rotation of the initial Cartesian reference frame and hence $\phi(\eta) = \phi(R^* \eta R)$

where R = the rotation matrix.

Further, a deformable medium is elastically isentropic if and only if $\phi(\eta)$ is a function of the three invariants $I_1$, $I_2$ and $I_3$ of $\eta$.

In reference [1] it is shown that:

$$\text{(B-3)} \qquad \frac{\partial I_1}{\partial \eta} = E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{(B-4)} \qquad \frac{\partial I_2}{\partial \eta} = I_1 E_3 - \eta$$

$$\text{(B-5)} \qquad \frac{\partial I_3}{\partial \eta} = \text{Co}\,\eta$$

It follows, that for an isotropic medium

$$\frac{\partial \phi}{\partial \eta} = \frac{\partial \phi}{\partial I_1}\frac{\partial I_1}{\partial \eta} + \frac{\partial \phi}{\partial I_2}\frac{\partial I_2}{\partial \eta} + \frac{\partial \phi}{\partial I_3}\frac{\partial I_3}{\partial \eta} \qquad \text{(Chain Rule)}$$

or $$\frac{\partial \phi}{\partial \eta} = \frac{\partial \phi}{\partial I_1} E_3 + \frac{\partial \phi}{\partial I_2}(I_1 E_3 - \eta) + \frac{\partial \phi}{\partial I_3}\,\text{Co}\,\eta$$

III. Assume the energy of deformation per unit initial volume is developed as a power series in terms of the three strain invariants $I_1$, $I_2$ and $I_3$. If $\eta$ is assumed infinitesimal, $I_1$ is of the order of $\epsilon$, $I_2$ of the order $\epsilon^2$, and $I_3$ of the order $\epsilon^3$. We can then say:

$$\text{(B-7)} \qquad \phi(\eta) = \phi(I_1, I_2, I_3) = A_0 + A_1 I_1 + \tfrac{1}{2} A_2 I_1^2 + \tfrac{1}{3\cdot 2} A_3 I_1^3 + \cdots$$
$$+ B_1 I_2 + B_2 I_2^2 + \cdots$$
$$+ C_1 I_3 + C_2 I_3^2 + \cdots + D_1 I_1 I_2 + \cdots$$

Neglecting terms of higher order than $\epsilon^3$ we have

$$\text{(B-8)} \qquad \phi(\eta) = A_0 + A_1 I_1 + \tfrac{1}{2} A_2 I_1^2 + \tfrac{1}{3\cdot 2} A_3 I_1^3 + B_1 I_2 + \frac{C_1}{2} I_1 I_2 + D_1 I_3$$

or in terms of $\phi = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \cdots$

where $\phi_0$ consists of terms independent of $\eta$

$\phi_1$ terms linear in elements of $\eta$

$\phi_2$ terms quadratic in elements of $\eta$ and hence a linear combination of $I_1^2$ and $I_2$, and

$\phi_3$ a linear combination of $I_1^3$, $I_1 I_2$ and $I_3$

(B-9) $$\phi = \phi_0 + \underbrace{A_1 I_1}_{\phi_1} + \underbrace{\left(\frac{A_2}{2} I_1^2 + B_1 I_2\right)}_{\phi_2} + \underbrace{\left(\frac{A_3 I_1^3}{3 \cdot 2} + \frac{C_1}{2} I_1 I_2 + D_1 I_3\right)}_{\phi_3}$$

Differentiating with respect to $\eta$, we have:

(B-10) $$\frac{\partial \phi}{\partial \eta} = A_1 \frac{\partial I_1}{\partial \eta} + A_2 I_1 \frac{\partial I_1}{\partial \eta} + B_1 \frac{\partial I_2}{\partial \eta} + \frac{A_3}{2} I_1^2 \frac{\partial I_1}{\partial \eta} + \frac{C_1}{2}\left(\frac{\partial I_1}{\partial \eta} I_2 + I_1 \frac{\partial I_2}{\partial \eta}\right) + D_1 \frac{\partial I_3}{\partial \eta}$$

Since $\frac{\partial I_1}{\partial \eta} = E_3$ ; $\frac{\partial I_2}{\partial \eta} = (I_1 E_3 - \eta)$ ; $\frac{\partial I_3}{\partial \eta} = Co\eta$

(B-11) $$\frac{\partial \phi}{\partial \eta} = A_1 E_3 + A_2 I_1 E_3 + B_1 (I_1 E_3 - \eta) + \frac{A_3}{2} I_1^2 E_3 + \frac{C_1}{2}\left[I_2 E_3 + I_1 (I_1 E_3 - \eta)\right] + D_1 \, Co\eta$$

Let

(B-12) $(A_2 + B_1) = \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ Lame's Constants ($\lambda$ & $\mu$)

(B-13) $B_1 = -2\mu = -2G$ ($\nu$ = Poisson's ratio)

(B-14) $\frac{A_3 + C_1}{2} = l$

(B-15) $C_1 = -2m$ l, m, and n = 3rd order constants

(B-16) $D_1 = n$

Substituting the elastic constants into $\frac{\partial \phi}{\partial \eta}$ we have;

(B-17) $\frac{\partial \phi}{\partial \eta} = A_1 E_3 + \lambda I_1 E_3 + 2G\eta + (l I_1^2 - 2m I_2) E_3 + 2m I_1 \eta + n \operatorname{co}\eta$

which is the 3rd order approximation obtained by Murnaghan [1].

The higher order relation between stress and strain resulting from the generalized relationship $T = \frac{\rho x}{\rho a} J \left(\frac{\partial \phi}{\partial \eta}\right) J^*$ is then:

(B-18) $T = \frac{\rho x}{\rho a} J \left[ A_1 E_3 + \lambda I_1 E_3 + 2G\eta + (l I_1^2 - 2m I_2) E_3 + 2m I_1 \eta + n \operatorname{co}\eta \right] J^*$

If the elastic medium is initially unstrained, $\frac{\rho x}{\rho a} = 1$ and $\eta$ is a zero or null matrix, therefore $J$ is a rotation matrix only. If $T_o$ is the initial stress in the unstrained condition, the above tensor equation becomes

(B-19) $T_o = A_1 E_3$

where $T_o$ is due to a hydrostatic pressure or tension.

For an isotropic medium, we can say $A_1 = -p_o$ (i.e. initial hydrostatic pressure) or $T_o = -p_o E_3$. With an initial hydrostatic pressure of $-p_o$, equation (B-18) becomes:

(B-20) $T = \frac{\rho x}{\rho a} J \left[ -p_o E_3 + \lambda I_1 E_3 + 2G\eta + (l I_1^2 - 2m I_2) E_3 + 2m I_1 \eta + n \operatorname{co}\eta \right] J^*$

For an isotropic medium, the coordinates of the stress tensor are furnished in the final rectangular Cartesian reference frame in which the coordinates of $JJ^*$ are furnished by the elements of $M = J^* J$ by the elements of the matrix $\frac{\rho x}{\rho a} M \frac{\partial \phi}{\partial \eta}$ where $J = R M^{1/2}$ and $J^* J = M = 2\eta + E_3$

Thus, the relation between stress and strain becomes:

(B-21) $$T' = \frac{\rho_x}{\rho_a}(E_3 + 2\eta)\Big[-p_0 E_3 + (\lambda I_1 E_3 + 2G\eta) + (l I_1^2 - 2m I_2)E_3 + 2m I_1 \eta + n\,\mathrm{co}\,\eta\Big]$$

$$T' = \frac{\rho_x}{\rho_a}\Big[-p_0 E_3 + \lambda I_1 E_3 + 2(G - p_0)\eta + (l I_1^2 - 2m I_2)E_3 + 2(m+\lambda)I_1\eta + n\,\mathrm{co}\,\eta + 4G\eta\eta\Big]$$

or for $p_0 = 0$

(B-22) $$T' = \frac{\rho_x}{\rho_a}\Big[\lambda I_1 E_3 + 2G\eta + (l I_1^2 - 2m I_2)E_3 + 2(m+\lambda)I_1\eta + n\,\mathrm{co}\,\eta + 4G\eta\eta\Big]$$

Murnaghan [1], has shown that

(B-23) $$\left(\frac{\rho_a}{\rho_x}\right)^2 = \det(E_3 + 2\eta) = (\det J)^2 = 1 + 2I_1 + 4I_2 + 8I_3$$

or $$\frac{\rho_x}{\rho_a} = (1 + 2I_1 + 4I_2 + 8I_3)^{-1/2}$$

To a second order approximation, the compression ratio is

(B-24) $$\frac{\rho_x}{\rho_a} = (1 + 2I_1 + 4I_2)^{-1/2} = 1 - I_1 + \left(\tfrac{3}{2} I_1^2 - 2I_2\right)$$

neglecting terms higher than 2nd order, we get

$$T' = \lambda I_1 E_3 + 2G\eta + (l I_1^2 - 2m I_2)E_3 + 2(m+\lambda)I_1\eta + n\,\mathrm{co}\,\eta + 4G\eta\eta - \lambda I_1^2 E_3 - 2G I_1 \eta$$ or

(B-25) $$T = \lambda I_1 E_3 + 2G\eta + \big[(l - \lambda)I_1^2 - 2m I_2\big]E_3 + 2(m + \lambda - G)I_1\eta + n\,\mathrm{co}\,\eta + 4G\eta\eta$$

## SUMMARY

Third-Order Approximation of stress-strain relation (three dimensions).

(B-25) $T = \lambda I_1 E_3 + 2G\eta + \left[(l-\lambda)I_1^2 - 2mI_2\right]E_3 + 2(m+\lambda-G)I_1\eta + n\,\mathrm{co}\,\eta + 4G\eta\eta$

where

| | | |
|---|---|---|
| $T$ | = | stress tensor |
| $\eta$ | = | strain tensor |
| $E_3$ | = | identity tensor |
| $I_1$ | = | First invariant of strain tensor |
| $I_2$ | = | Second invariant of strain tensor |
| $\lambda, G$ | = | Second order elastic constants (elastic constants of Lame) |
| $l, m, n$ | = | Third Order Elastic Constants |
| $\lambda$ | = | $\frac{\nu E}{(1+\nu)(1-2\nu)} = \frac{2G\nu}{1-2\nu}$ |
| $\nu$ | = | Poisson's ratio $= \frac{\lambda}{2(\lambda+G)}$ |
| $E$ | = | Young's Modulus $= \frac{G(3\lambda+2G)}{\lambda+G} = 2G(1+\nu)$ |
| $G$ | = | $\frac{E}{2(1+\nu)}$ = shear modulus |

## APPENDIX C

## DERIVATION OF CLASSICAL EQUILIBRIUM EQUATIONS

### I. Balance of Forces

Considering the elemental plate shown in figures C-1 and C-2, we have for the x and y component forces

x component

(C-1) $$\left(N_x + \frac{\partial N_x}{\partial x} dx\right) dy - N_x\, dy + \left(N_{yx} + \frac{\partial N_{yx}}{\partial y} dy\right) dx - N_{yx} dx + X\, dx\, dy = 0$$

$$= \frac{\partial N_x}{\partial x} dx\, dy + \frac{\partial N_{yx}}{\partial y} dx\, dy + X\, dx\, dy = 0$$

$$= \frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} + X = 0$$

where $X$ is the x body or tangential force component per unit area of the middle plane of the plate

y component

Similarly we have for the y component forces

(C-2) $$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + Y = 0$$

If we assume $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are small compared to $dx$ and $dy$, we have for the projection of the $N_x$ forces on the $z$ axis:

(C-3) $$\left(N_x + \frac{\partial N_x}{\partial x} dx\right)\left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx\right) dy - N_x \frac{\partial w}{\partial x}$$

$$= \frac{\partial N_x}{\partial x} \frac{\partial w}{\partial x} dx\, dy + N_x \frac{\partial^2 w}{\partial x^2} dx\, dy + \frac{\partial N_x}{\partial x} \frac{\partial^2 w}{\partial x^2} d^2x\, dy$$

or, reducing to second order terms, we have

(C-4) $$N_x \frac{\partial^2 w}{\partial x^2} dx\,dy$$

and similarly, the $z$ component of the $N_y$ forces becomes

(C-5) $$N_y \frac{\partial^2 w}{\partial y^2} dx\,dy$$

since $N_{xy} \cong N_{yx}$ from equilibrium of moments about the $z$ axis the $z$ component of the $N_{xy}$ and $N_{yx}$ forces is

(C-6) $$2\,N_{xy} \frac{\partial^2 w}{\partial x \partial y}$$

Thus, the membrane portion of the $z$ component forces become

(C-7) $$N_x \frac{\partial^2 w}{\partial x^2} + 2\,N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}$$

The effect of shears $Q_x$ and $Q_y$, as shown in figure C-3 also contribute to the equilibrium of $z$ component forces, and can be shown to be: $\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\right) dx\,dy$

The lateral load component is $p\,dx\,dy$ and the body force components are $-X \frac{\partial w}{\partial x} dx\,dy$ and $-Y \frac{\partial w}{\partial y} dx\,dy$

Thus, the combination of $z$ components in the general case is:

(C-8) $$p + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} - X \frac{\partial w}{\partial x} - Y \frac{\partial w}{\partial y} = 0$$

## BALANCE OF MOMENTS

Considering figure C-3, by taking moments about the x axis and considering the right hand rule for positive moments, we have

(C-9) $$\left(M_{xy} + \frac{\partial M_{xy}}{\partial x} dx\right) dy - M_{xy} dy + M_y dx - \left(M_y + \frac{\partial M_y}{\partial y} dy\right) dx + \left(Q_y + \frac{\partial Q_y}{\partial y} dy\right) dx\,dy = 0$$

Simplifying and retaining terms up to the second order only, we have

(C-10) $$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0$$

Similarly, by taking moments about the y axis, we have

(C-11) $$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x = 0$$

Differentiating the 1st equation with respect to y and the 2nd with respect to x and adding, we have:

(C-12) $$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{yx}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial x \partial y} - \frac{\partial Q_y}{\partial y} - \frac{\partial Q_x}{\partial x} = 0$$

or since $M_{yx} = -M_{xy}$ $(\tau_{xy} = \tau_{yx})$

we have

(C-13) $$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial x \partial y} = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}$$

But from the $z$ component force equation, we have

(C-14) $$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -\left(p + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} - X\frac{\partial w}{\partial x} - Y\frac{\partial w}{\partial y}\right)$$

Thus we have

(C-15) $$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\left(p + N_x\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y\frac{\partial^2 w}{\partial y^2} - X\frac{\partial w}{\partial x} - Y\frac{\partial w}{\partial y}\right)$$

The moments resulting from the stresses distributed on the x and y faces of the differential element shown in figure C-3 are

(C-16) $$M_x = \int_{-t/2}^{t/2} \sigma_x \, z \, dz$$

(C-17) $$M_y = \int_{-t/2}^{t/2} \sigma_y \, z \, dz$$

(C-18) $$M_{xy} = -\int_{-t/2}^{t/2} \tau_{xy} \, z \, dz$$

From the plane stress equations of the linearized Hooke's law, we have

(C-19) $$e_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

(C-20) $$e_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

(C-21) $$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

or, in terms of stresses, we have

(C-22) $$\sigma_x = \frac{E}{1-\nu^2}(e_x + \nu e_y)$$

(C-23) $$\sigma_y = \frac{E}{1-\nu^2}(e_y + \nu e_x)$$

(C-24) $$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}$$

since

$$e_x = \frac{z}{r_x} = -z\frac{\partial^2 w}{\partial x^2}$$

$$e_y = \frac{z}{r_y} = -z\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

by substituting the stress relations (C-22), (C-23) and (C-24) in terms of $z$ and the curvatures into equations (C-16), (C-17) and (C-18) and integrating, we have:

(C-25) $$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

(C-26) $$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

(C-27) $$M_{xy} \simeq D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

where D is the flexural rigidity $D \equiv \dfrac{Et^3}{12(1-\nu^2)}$

Differentiating equations (C-25), (C-26), and (C-27), we have

(C-28) $$\frac{\partial^2 M_x}{\partial x^2} = -D\left(\frac{\partial^4 w}{\partial x^4} + \nu\frac{\partial^4 w}{\partial x^2 \partial y^2}\right)$$

(C-29) $$\frac{\partial^2 M_y}{\partial y^2} = -D\left(\frac{\partial^4 w}{\partial y^4} + \nu\frac{\partial^4 w}{\partial x^2 \partial y^2}\right)$$

(C-30) $$\frac{\partial^2 M_{xy}}{\partial x \partial y} = D(1-\nu)\frac{\partial^4 w}{\partial x^2 \partial y^2}$$

Thus

(C-31) $$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -D\left(\frac{\partial^4 w}{\partial x^4} + 2\nu\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + 2(1-\nu)\frac{\partial^4 w}{\partial x^2 \partial y^2}\right) = -D\nabla^4 w$$

Finally, substituting equation (C-31) into equation (C-15) we have

$$\text{(C-32)} \quad \nabla^4 w = \frac{p}{D} + \frac{1}{D}\left( N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} - X \frac{\partial w}{\partial x} - Y \frac{\partial w}{\partial y} \right)$$

From the Stress Function F defined as

$$\frac{\partial^2 F}{\partial y^2} = \sigma_x = \frac{N_x}{t}$$

$$\frac{\partial^2 F}{\partial x^2} = \sigma_y = \frac{N_y}{t}$$

$$-\frac{\partial^2 F}{\partial x \partial y} = \tau_{xy} = \frac{N_{xy}}{t}$$

We have

$$\text{(C-33)} \quad \nabla^4 w = \frac{p}{D} - \frac{t}{D}\left( \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{X}{t}\frac{\partial w}{\partial x} - \frac{Y}{t}\frac{\partial w}{\partial y} \right)$$

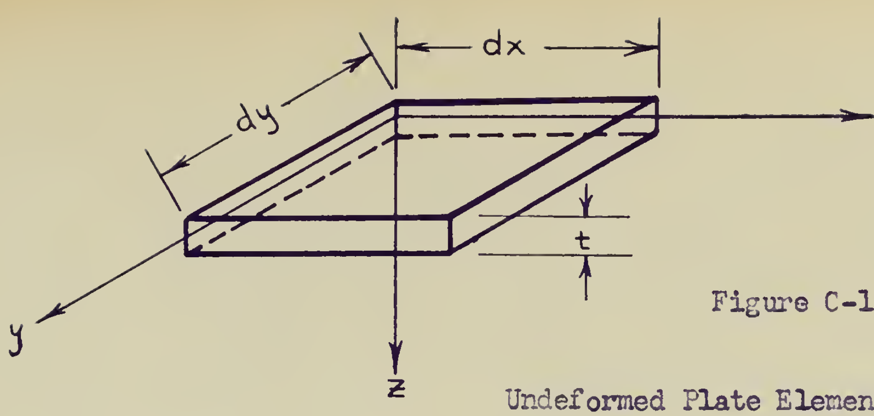


Figure C-1

Undeformed Plate Element

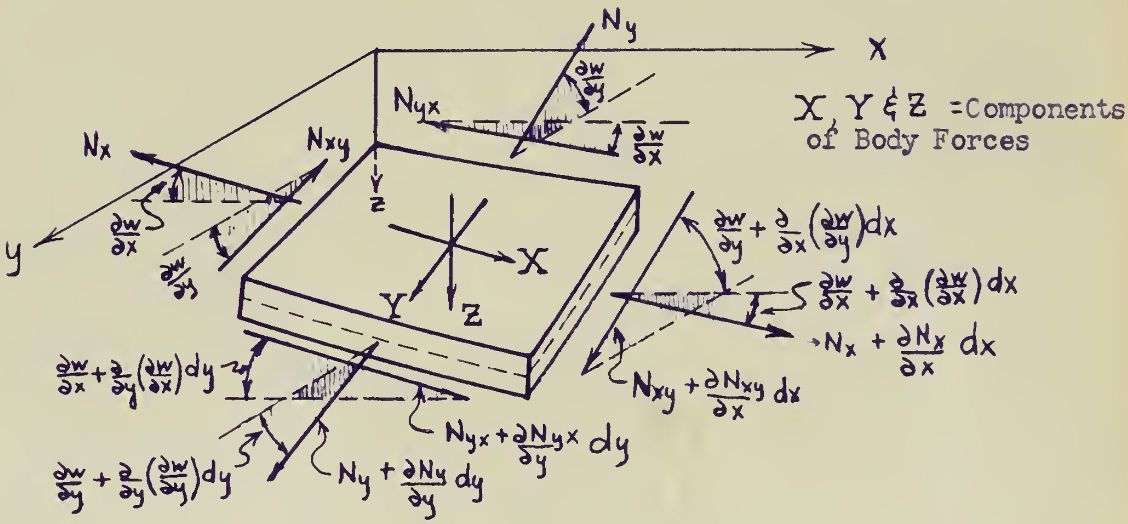


Figure C-2

Deformed Plate Element Showing Forces and Slopes ( Greatly Exaggerated )

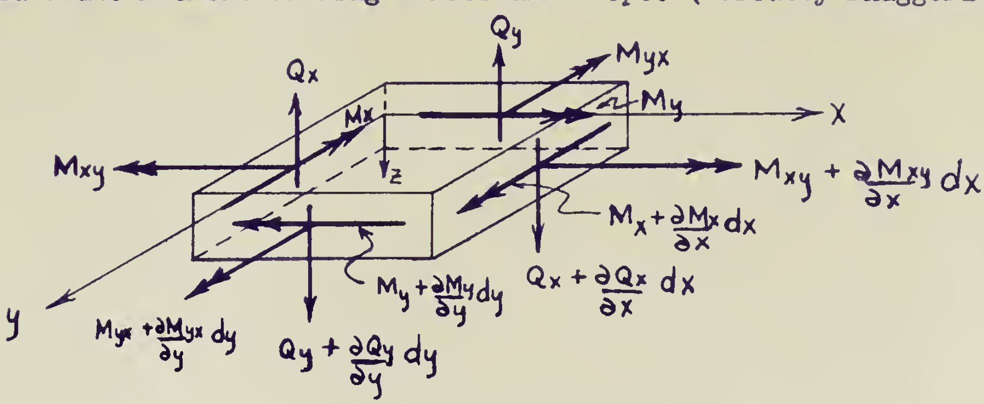

Figure C-3

Sub-Element Showing Moments and Shear